## 2.1 MATRIX OPERATIONS

$$
\begin{array}{cc}
 & \begin{array}{ccccc} & & \text{Column } j & & \end{array} \\
\text{Row } i & \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} = A \\
 & \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array}
\end{array}
$$

FIGURE 1 Matrix notation

The **diagonal entries** in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \ldots$ and they form the **main diagonal** of $A$.

A **diagonal matrix** is a square matrix whose nondiagonal entries are zero.

An $m \times n$ matrix of zeros is a **zero matrix** and is written as **0**.

### Sums and Scalar Multiples

Matrix addition and scalar multiplication works like you would expect it to. Add terms in the same position. Multiply the scalar by all terms in the matrix.

$$\begin{pmatrix} 3 & 0 & 5 \\ -1 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 6 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 7 \\ 5 & 8 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix} = \text{DOES NOT EXIST}$$

$$3\begin{pmatrix} 1 & 1 & 2 \\ 6 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 6 \\ 18 & 15 & 21 \end{pmatrix}$$

THEOREM 1

Let $A$, $B$, and $C$ be matrices of the same size, and let $r$ and $s$ be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = \text{rA} + \text{rB}$

e. $(r + s)A = \text{rA} + \text{sA}$

f. $r(\text{sA}) = (\text{rs})A$

### Matrix Multiplication does not work quite how you might expect it to!

GOAL:

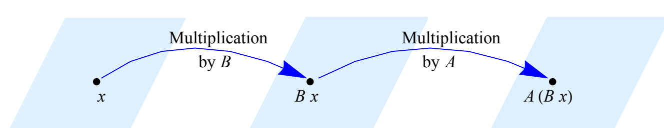


FIGURE 2: Multiplication by $B$ and then $A$

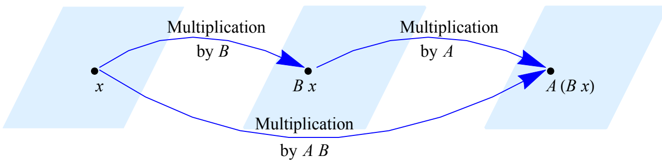


FIGURE 3: Multiplication by $A\,B$

DEFINITION : $$A\,B = \left[A\,\boldsymbol{b}_1\;A\,\boldsymbol{b}_2\cdots A\,\boldsymbol{b}_p\right]$$

*Multiplication of matrices corresponds to composition of linear transformations.*

Compute $A\,B$, where $A = \begin{pmatrix} 2 & 5 & -2 \\ -1 & -5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 1 & 2 \end{pmatrix}$.

SOLUTION Write $B = [\,\boldsymbol{b}_1 \quad \boldsymbol{b}_2\,]$, and compute:

$$A\,\boldsymbol{b}_1 = \begin{pmatrix} 2 & 5 & -2 \\ -1 & -5 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, \qquad A\,\boldsymbol{b}_2 = \begin{pmatrix} 2 & 5 & -2 \\ -1 & -5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

$$= \begin{pmatrix} 31 \\ -28 \end{pmatrix} \qquad\qquad = \begin{pmatrix} 10 \\ -10 \end{pmatrix}$$

$$\text{Then} \quad A\,B = \left[\,A\,\boldsymbol{b}_1 \quad A\,\boldsymbol{b}_2\,\right] = \begin{pmatrix} 31 & 10 \\ -28 & -10 \end{pmatrix}$$

Each column of $A\,B$ is a linear combination of the columns of $A$ using weights from the corresponding column of $B$.

The number of columns of $A$ must equal the number of rows in $B$ in order for $A\,B$ to exist.

Row - Column Rule for Computing $A\,B$

$$(A\,B)_{i\,j} = a_{i\,1}\,b_{1\,j} + a_{i\,2}\,b_{2\,j} + \cdots + a_{i\,n}\,b_{n\,j} = \sum_{k=1}^{n} a_{ik}\,b_{kj}$$

Row View :

$$\text{row}_i\,(A\,B) = \text{row}_i\,(A)\cdot B$$

**EX5**

row of AB ○ 1 ○ 2 ○ 3 ● 4 ○ 5

column of AB ○ 1 ○ 2 ● 3

$$\overset{A}{\begin{pmatrix} 3 & 5 & -1 & 2 \\ -2 & 1 & -4 & 1 \\ 0 & 3 & 0 & -4 \\ -4 & -3 & -2 & 4 \\ -2 & 1 & 0 & 0 \end{pmatrix}} \overset{B}{\begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & 2 \\ 4 & 1 & 3 \\ 2 & 0 & 6 \end{pmatrix}}$$

$$AB = \begin{pmatrix} 6 & -11 & 22 \\ -18 & -6 & -6 \\ -8 & -6 & -18 \\ -8 & 4 & 8 \\ -4 & -2 & 0 \end{pmatrix}$$

$$(-4)(1) + (-3)(2) + (-2)(3) + (4)(6)$$
$$=$$
$$-4 + -6 + -6 + 24$$
$$=$$
$$8$$

Number of rows for A 4

Number of columns for A 4

new matrices

$$A= \begin{pmatrix} 1 & -4 & 4 & -2 \\ -2 & -1 & -5 & 2 \\ 3 & -2 & -3 & -3 \\ 1 & 0 & -5 & -2 \end{pmatrix} \quad B= \begin{pmatrix} -1 & -5 & -5 & -5 \\ -3 & 3 & -2 & -5 \\ 4 & 5 & 0 & -5 \\ -1 & 0 & -3 & 0 \end{pmatrix}$$

$$AB= \begin{pmatrix} 29 & 3 & 9 & -5 \\ -17 & -18 & 6 & 40 \\ -6 & -36 & -2 & 10 \\ -19 & -30 & 1 & 20 \end{pmatrix} \quad BA= \begin{pmatrix} -11 & 19 & 61 & 17 \\ -20 & 13 & 4 & 28 \\ -11 & -21 & 16 & 12 \\ -10 & 10 & 5 & 11 \end{pmatrix}$$

THEOREM 2

Let $A$ be an $m \times n$ matrix, and let $B$ and $C$ have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar $r$

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

f. $AB \neq BA$ MOST of the time.

Powers of a Matrix

$$A^k = \underbrace{A\,A\cdots A}_{k\text{ times}}$$

Note: $A^0 = I$

**The Transpose of a Matrix**

Let A be a matrix. The transpose, denote $A^T$, is formed by writing the rows of A down the columns of $A^T$

$$A = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}$$

$$A^T = \begin{pmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{pmatrix}$$

THEOREM 3

Let $A$ and $B$ denote matrices whose sizes are appropriate for the following sums and products.

a. $\left((A^T)^T\right) = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar $r$, $(r\,A)^T = r\left(A^T\right)$

d. $(A\,B)^T = B^T\,A^T$