The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

The **null space** of a matrix A is the set Nul A of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

A spanning set can be too big.

A linearly independent set can be too small.

A **basis** is just right.

DEFINITION

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

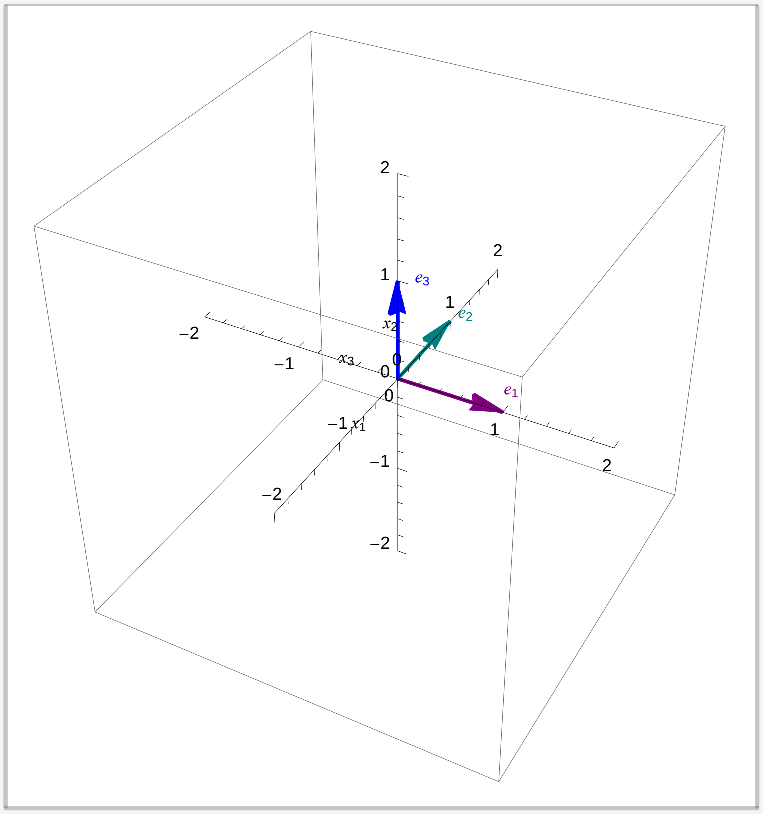

FIGURE 3: The standard basis

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

The set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$.

THEOREM 13

The pivot columns of a matrix A form a basis for the column space of A.

Warning Be careful to use *pivot columns of A itself* for the basis of Col A. The columns of an echelon form B are often not in the column space of A.

To find a basis for the null space, use the vectors from the parametric vector form of the solution for $A\mathbf{x} = \mathbf{0}$.

EXAMPLE: Find a basis for Col A and Nul A, when

$$A = \begin{pmatrix} 1 & 0 & 3 & 2 & 1 & 9 \\ 2 & 0 & 6 & 5 & 2 & 8 \\ 3 & 0 & 9 & 7 & 3 & 7 \\ 4 & 0 & 12 & 8 & 8 & 8 \end{pmatrix}$$

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TraditionalForm[RowReduce[A]]
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$$\begin{pmatrix} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A basis for Col A is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 7 \\ 8 \end{pmatrix} \right\}$

To find a basis for Nul A, pretend (or actually put in) there is a right hand side of all zeros and write down the parametric form for your solution

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ x_1, x_4, x_5, x_6 are basic, and x_2, x_3 are free variables

$x_1 = -3x_3$
$x_2 = x_2$
$x_3 = x_3$
$x_4 = 0$
$x_5 = 0$
$x_6 = 0$

$$x = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A basis for Nul A is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

FACT: Any two bases have the same number of vectors.

DEFINITION

The **dimension** of a nonzero subspace H is the number of vectors in any basis for H. The dimension of the zero subspace {**0**} is defined to be zero.

A 3-D subspace of R^3 is equal to R^3

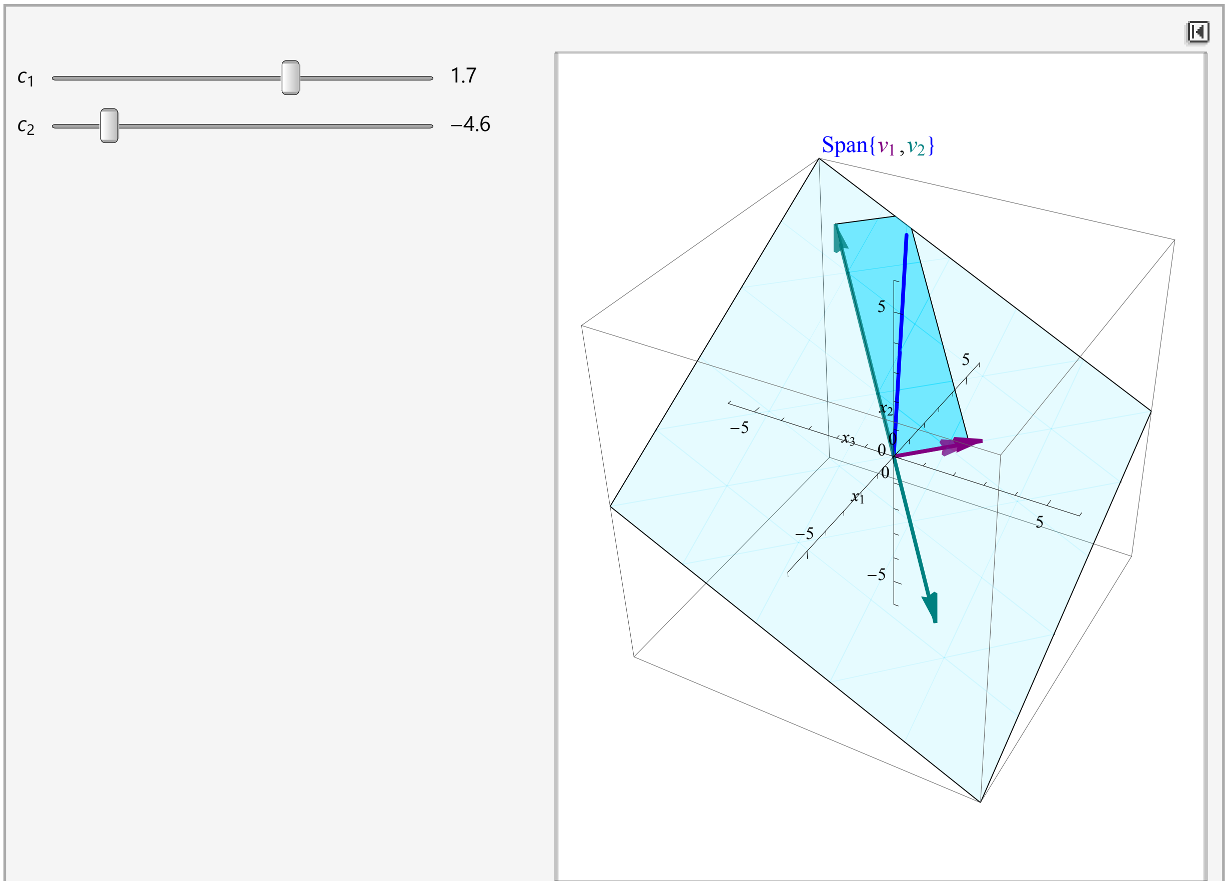

FIGURE 1: Span{v_1,v_2} as a plane through the origin

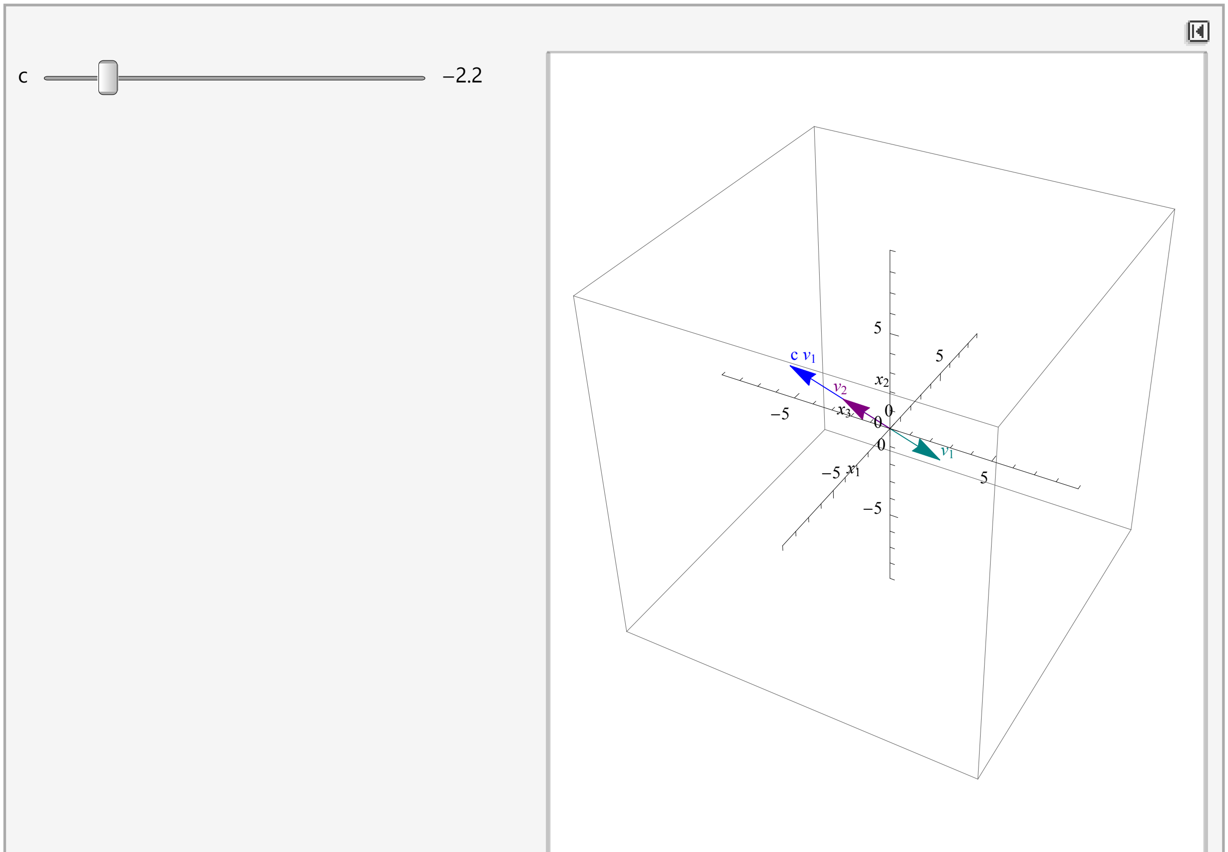

Figure 1A: Span{v_1,v_2} when v_2 = k v_1

DEFINITION

The **rank** of a matrix A is the dimension of the column space of A.

Every column of a matrix is either a pivot (contributing to the dimension of Col A) or it corresponds to a free variable (contributing to the dimension of Nul A).

THEOREM 14

The Rank Theorem
If a matrix A has n columns, then rank A + dim Nul $A = n$

Question: What is the dim Nul A if A is a 5x7 matrix with rank 4?

ANSWER: 7-4=3

ANY TWO WILL DO!

Linearly independent and spanning === basis

S is a spanning set with the right number (dimension) of vectors === basis

S is a linearly independent set with the right number (dimesion) of vectors === basis.

QUESTION: Suppose a subspace H has dimension 3. If S is a linearly independent subset of H with 2 vectors, is S a basis for H?

NO - S is toooo small.

QUESTION: Suppose H is a five dimensional subspace, and S is a set of five vectors that are all linearly independent. Is S a basis for H?

YES - S has the right number of vectors and is linearly independent.

THEOREM 16

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul $A = \{\mathbf{0}\}$

r. dim Nul $A = 0$