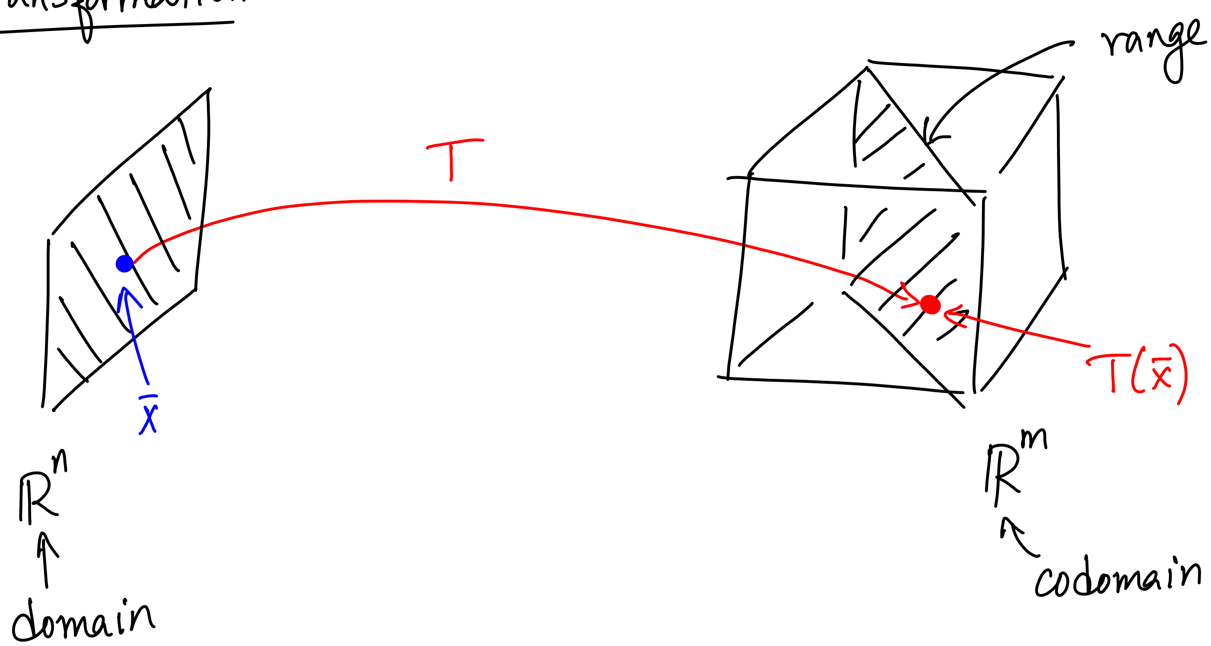
MATH 220 - Lecture 10 (09/19/2013)

Transformations



T($\bar{x}$) is the image of $\bar{x}$ under transformation T. The set of all images under T is called the range of T.

$$T: \underset{\text{domain}}{\mathbb{R}^n} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^m} \qquad \text{or} \qquad \bar{x} \longmapsto T(\bar{x})$$

Matrix Transformation

$T(\bar{x}) = A\bar{x}$ or $\bar{x} \longmapsto A\bar{x}$ for $A \in \mathbb{R}^{m\times n}$.

Prob 5, pg 68 $\longrightarrow$ continued from Lecture 8...

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

Find $\bar{x}$ such that $T(\bar{x}) = A\bar{x} = \bar{b}$. Is this $\bar{x}$ unique?

We will use the results on solutions of systems of the form $A\bar{x} = \bar{b}$ to answer questions of this form.

Reword: Find a solution to $A\bar{x}=\bar{b}$. Is the solution unique?

$$\left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{array}\right] \xrightarrow{R_2+3R_1} \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{array}\right] \xrightarrow{R_2 \times -1/8} \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$x_3 \to$ free

$$\xrightarrow{R_1+5R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$x_1=3$, $x_2=1$ or $\bar{x}=\begin{bmatrix}3\\1\\0\end{bmatrix}$ is one $\bar{x}$ such that $T(\bar{x})=\bar{b}$. It is not unique.

A is 2×3, so $T(\bar{x})=A\bar{x} : \mathbb{R}^3 \to \mathbb{R}^2$.

$$\left.\begin{array}{l} x_1 = 3-3x_3 \\ x_2 = 1-2x_3 \end{array}\right\} \quad \text{So} \quad \bar{x} = \begin{bmatrix}3\\1\\0\end{bmatrix} + \begin{bmatrix}-3\\-2\\1\end{bmatrix}s, \quad s\in\mathbb{R}.$$

All the $\bar{x}$ (as described above) have $T(\bar{x})=A\bar{x}=\bar{b}$.

Linear Transformations

$T:\mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if

(i) $T(\bar{u}+\bar{v}) = T(\bar{u})+T(\bar{v})$ for all $\bar{u},\bar{v}\in\mathbb{R}^n$, and

(ii) $T(c\bar{u}) = cT(\bar{u})$ for $\bar{u}\in\mathbb{R}^n$, $c\in\mathbb{R}$.

"in" or "element of"

c is a scalar

In words, we say that

T preserves vector addition and scalar multiplication.

(ii) immediately implies that

(iii) $T(\bar{0}) = \bar{0}$. as $T(0 \cdot \bar{u}) = 0 \cdot T(\bar{u}) = \bar{0}$ for any $\bar{u}$.

Combining (i) and (ii), we get

(iv) $T(\underbrace{c\bar{u}+d\bar{v}}_{\text{linear combination}}) = cT(\bar{u}) + dT(\bar{v})$ for $\bar{u}, \bar{v} \in R^n$ and $c, d \in R$ (or scalars).

We could specify (iv) alone as the definition of a linear transformation.

Prob 17, pg 68

$T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear. It is given that

for $\bar{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \bar{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $T(\bar{u}) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Find $T(2\bar{u}), T(3\bar{v})$, and $T(2\bar{u}+3\bar{v})$.

$$T(2\bar{u}) = 2T(\bar{u}) = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

$$T(3\bar{v}) = 3T(\bar{v}) = 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

$$T(2\bar{u}+3\bar{v}) = T(2\bar{u}) + T(3\bar{v}) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

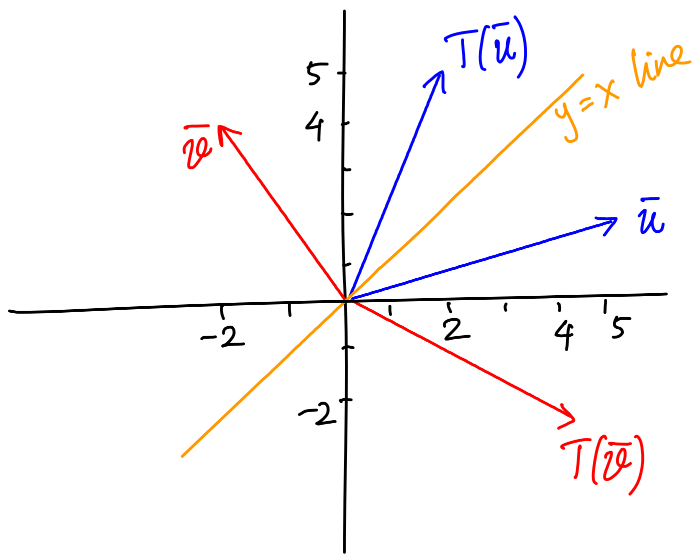
Equivalently, $T(2\bar{u}+3\bar{v}) = 2T(\bar{u}) + 3T(\bar{v}) = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$.

Prob 15, pg 68

$\bar{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. $T(\bar{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Plot $\bar{u}, \bar{v}, T(\bar{u}), T(\bar{v})$

$$T(\bar{u}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$T(\bar{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



Notice that T here flips the x- and y-coordinates. Equivalently, T reflects the vectors through the $y=x$ line ($45°$ line).

This problem illustrates the applications of linear transformations in image (and video) analysis. For instance, when you rotate a picture that you took with the camera set sideways so that the subjects in the picture are upright involves the application of a linear transformation.

We will discuss more such problems in the next lecture.

We now consider an example that illustrates how we can define the general form of an LT (linear transformation), once we have defined the images of certain canonical vectors under the same LT. Taking a lead from this example, we will define next the matrix of any linear transformation.

Prob 19, pg 68

Let $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under $\underset{\text{linear transformation}}{T} : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\bar{e}_1) = \bar{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and

$T(\bar{e}_2) = \bar{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right)$ and $\underbrace{T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}_{\text{the image of any vector under } T}$.

Since T is a linear transformation, $T(c\bar{u} + d\bar{v}) = cT(\bar{u}) + dT(\bar{v})$.

$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$c\bar{e}_1 + d\bar{e}_2$ for

$c = 5$, $d = -3$

$$T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) = 5T(\bar{e}_1) + (-3)T(\bar{e}_2)$$

$$= 5\underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{\bar{y}_1} + (-3)\underbrace{\begin{bmatrix} -1 \\ 6 \end{bmatrix}}_{\bar{y}_2} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}.$$

Similarly, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\bar{e}_1} + x_2\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\bar{e}_2} = x_1\bar{e}_1 + x_2\bar{e}_2$

Hence $T(\bar{x}) = x_1\underbrace{T(\bar{e}_1)}_{\bar{y}_1} + x_2\underbrace{T(\bar{e}_2)}_{\bar{y}_2} = x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$.

Check: $T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 2(5) - (-3) \\ 5(5) + 6(-3) \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$.

Matrix of a Linear Transformation

Given a LT $T: \mathbb{R}^n \to \mathbb{R}^m$, we can write

$T(\bar{x}) = A\bar{x}$, where $A = [T(\bar{e}_1)\ T(\bar{e}_2) \cdots T(\bar{e}_n)]$ where

$$\bar{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ entry.}$$

$\bar{e}_j$ is the j^{th} unit vector.

In the above example, $A = [\bar{y}_1\ \bar{y}_2] = \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix}$.