MATH 220 – Lecture 13 (10/01/2013)

Review: Practice midterm

Prob 3: $\bar{u} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$, $\bar{w} = \begin{bmatrix} 0.5 \\ 2 \\ -5 \end{bmatrix}$, $\quad 3\bar{u} - \bar{v} = 2\bar{w}$.

$$A = [\bar{v} \ \bar{u} \ \bar{w}] = \begin{bmatrix} 2 & 1 & 0.5 \\ 5 & 3 & 2 \\ 7 & -1 & -5 \end{bmatrix}$$

$A\bar{x} = \bar{0}$ is the same as $\bar{v}x_1 + \bar{u}x_2 + \bar{w}x_3 = \bar{0}$,

where $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $\quad 3\bar{u} - \bar{v} = 2\bar{w}$ gives $\underset{x_1}{-\bar{v}} + \underset{x_2}{3\bar{u}} - \underset{x_3}{2\bar{w}} = \bar{0}$

So, $x_1 = -1$, $x_2 = 3$, $x_3 = -2$ is a nontrivial solution,

or $\bar{x} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ is a nontrivial solution.

The idea here is to rewrite the given relation so that it corresponds to the equation in question. As such, we could directly read off a nontrivial solution.

④. $\bar{b} = \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}$. $A \in \mathbb{R}^{3\times 3}$ such that $\bar{b} \notin \text{span}\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$, where $A = [\bar{a}_1\, \bar{a}_2\, \bar{a}_3]$.

Start with $A = \begin{bmatrix} 8 & 8 & 8 \\ -3 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$, as the last row of this matrix does not have a pivot, while the 3rd entry in $\bar{b}$ is nonzero $(=1)$.

$$A \xrightarrow{R_3 + R_2} \begin{bmatrix} 8 & 8 & 8 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} = A$$

Or, start with $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and then $\xrightarrow[R_3 + R_1]{R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

You could also, for example, use $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Just demonstrate that $A\bar{x} = \bar{b}$ is indeed inconsistent.

$$\left[\begin{array}{ccc|c} 5 & 5 & 5 & 8 \\ 5 & 5 & 5 & -3 \\ 5 & 5 & 5 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 5 & 5 & 5 & 8 \\ 5 & 5 & 5 & -3 \\ 0 & 0 & 0 & -7 \end{array}\right]$$

$\longrightarrow$ inconsistent system!

⑤ $\bar{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $\bar{v}_4 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

(a) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ span $\mathbb{R}^3$?

$$A = [\bar{v}_1\ \bar{v}_2\ \bar{v}_3\ \bar{v}_4] = \begin{bmatrix} 3 & 6 & 5 & 5 \\ 1 & 2 & -2 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[\substack{\text{and then} \\ R_2 - 3R_1 \\ R_3 - 4R_1}]{R_1 \rightleftarrows R_2}$$

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 11 & 5 \\ 0 & -7 & 9 & 2 \end{bmatrix} \xrightarrow{R_2 \rightleftarrows R_3} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -7 & 9 & 2 \\ 0 & 0 & 11 & 5 \end{bmatrix}$$

pivot in every row. So columns span $\mathbb{R}^3$.

(b) Does $\{\bar{v}_1, \bar{v}_2\}$ span $\mathbb{R}^3$?

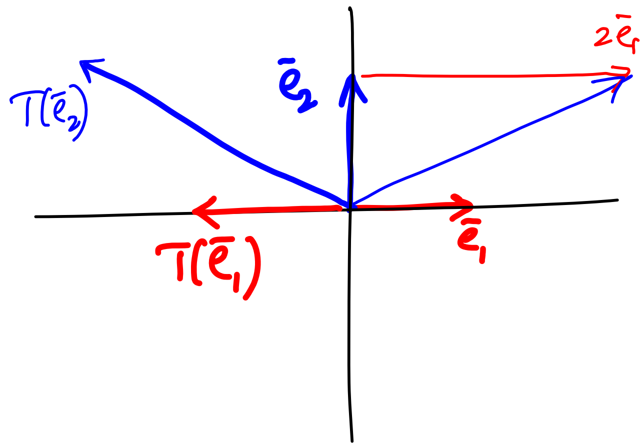
No! We need three pivots, but $A = [\bar{v}_1\ \bar{v}_2]$ can have at most two pivots.

Similarly, $\{\bar{v}_1, \bar{v}_2\}$ does not span $\mathbb{R}^2$, as vectors in $\mathbb{R}^2$ have 2 entries each, and cannot be written as linear combinations of $\bar{v}_1$ and $\bar{v}_2$, which have 3 entries each.

For the same reason, $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ cannot span $\mathbb{R}^4$.

(6) $T(\bar{e}_2) = \bar{e}_2 + 2\bar{e}_1$

and then reflect on vertical axis



$$T(\bar{e}_1) = -\bar{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(\bar{e}_2) = \bar{e}_2 - 2\bar{e}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$A = \left[T(\bar{e}_1) \ T(\bar{e}_2) \right] = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

(8) T/F

(a) T. See solutions. The reduced echelon form is unique.

(b) T. There is a pivot in each of the two columns so there are no free variables.

(c) F. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $T(\bar{x}) = A\bar{x}$ is both 1-to-1 and onto.

(d) $T : \mathbb{R}^n \to \mathbb{R}^m$ is an LT, then $T(\bar{x}) = A\bar{x}$ where $A \in \mathbb{R}^{m \times n}$

domain

codomain

F. $\mathbb{R}^m$ is the codomain.

(7) $$[A|b] = \left[\begin{array}{cc|c} 1 & 3 & k \\ 1 & -h & 2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 3 & k \\ 0 & -h-3 & 2-k \end{array}\right]$$

(a) inconsistent if $-h-3=0$ and $2-k \neq 0$, i.e.,

$$\boxed{h=-3,\ k \neq 2}.$$

(b) unique solution if $-h-3 \neq 0$, i.e., $\boxed{h \neq -3}$.

(c) infinitely many solutions when $-h-3=0$ and $2-k=0$, i.e., $\boxed{h=-3,\ k=2}$.