MATH 220 - Lecture 19 (10/22/2013)

Next week: Tuesday - Prof McDonald
Thursday - no class

Homework: Section 2.2 is due this Thursday (Oct 24)
2.3 will be due Wednesday (Oct 30)

The invertible matrix theorem

For an $n \times n$ matrix A, the following statements are equivalent.

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

$[A|I] \xrightarrow{EROs} [I|A^{-1}]$

(d) $A\bar{x}=\bar{0}$ has only trivial solution.

(e) Columns of A are LI.

(f) The LT $\bar{x} \mapsto A\bar{x}$ is one-to-one

pivot in every column

(g) $A\bar{x}=\bar{b}$ has a unique solution for every $\bar{b} \in \mathbb{R}^n$. pivot in each row & each column

(h) Columns of A span $\mathbb{R}^n$.

(i) The LT $\bar{x} \mapsto A\bar{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

pivot in every row

(j) The exists $C \in \mathbb{R}^{n \times n}$ such that $CA = I_n$. $C = A^{-1}$ works

(k) The exists $D \in \mathbb{R}^{n \times n}$ such that $AD = I_n$. $D = A^{-1}$ works

(l) A^T is invertible. $(A^{-1})^T = (A^T)^{-1}$

Prob 4 and 6, pg 115

Determine if matrix is invertible, using as few operations as possible.

4. $\begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$ There cannot exist $n=3$ pivots (because of the zero row). So matrix is not invertible.

6. $\begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ -3 & 6 & 0 \end{bmatrix} \xrightarrow{R_3+3R_1} \begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ 0 & -3 & -18 \end{bmatrix} \xrightarrow{R_3+\frac{3}{4}R_2} \begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ 0 & 0 & -63/4 \end{bmatrix}$

$n=3$ pivots. So matrix is invertible.

We could not guess that the matrix is invertible without some EROs here.

16. If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent. Explain why.

If A is invertible, then A^T is also invertible (by IMT).

invertible matrix theorem

So A^T has a pivot in every column, as it has n pivots. So, the columns are LI.

For problems in this section, it is not sufficient to bluntly state the IMT as the reason for your conclusions. You need to specify the details of why the conclusion follows. For the same reason, you should not try to memorize the index letters ((b), (f), or (j), for instance) in the IMT!

19.3

22. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

E and F commute means $EF = FE$. In this case, you want to show $EF = FE = I$.

Since $EF = I$, by IMT both E and F are invertible (statements (j) and (k) in IMT).

Since E is invertible, and $EF = I$, $F = E^{-1}$.

So $FE = E^{-1}E = I$

27. Let A and B be $n \times n$ matrices. Show that if AB is invertible, so is A. You cannot use Theorem 6(b), because you cannot *assume* that A and B are invertible. [*Hint:* There is a matrix W such that $ABW = I$. Why?]

AB is invertible. By IMT there exist nxn matrices C and D such that

$CAB = I$ and

$ABD = I$.

Hence $(CA)B = I$ and $A(BD) = I$. So there exist nxn matrices $E = CA$ and $F = BD$ such that $EB = I$, $AF = I$.

So both A and B are invertible by the IMT.

31. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Without using Theorems 5 or 8, explain why each equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.

IMT

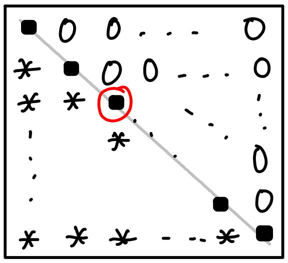
If A is invertible, $A\bar{x}=\bar{b}$ has a unique solution $\bar{x}=A^{-1}\bar{b}$

Of course, the result you want to prove here is implied by the theorems you are supposed to "avoid". The idea is to argue, from scratch, rather than just state "follows from IMT", for instance.

Since $A\bar{x}=\bar{b}$ has atleast a solution for every $\bar{b} \in \mathbb{R}^n$, columns of A span $\mathbb{R}^n$. Hence A has a pivot in every row. But A is $n \times n$, so A has exactly n pivots, and also a pivot in every column. Hence $A\bar{x}=\bar{b}$ has exactly one solution for every $\bar{b} \in \mathbb{R}^n$.

19.5

14. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.



The $n \times n$ matrix is invertible when all the n entries in the diagonal are nonzero. In this case, we can use each of these nonzero entries to zero out the entries below the diagonal in each column.

Hence, we will get n pivots, which makes the matrix invertible by IMT.

Notice that if a diagonal entry is zero, we will not get n pivots.