# MATH 220 - Lecture 23 (11/05/2013)

Homework on Section 2.8 due Friday, Nov 8.
Section 2.9 due Tuesday, Nov 12.

Recall (results on bases of subspaces, dimension, rank, etc.)

Basis of a subspace $H$ is an LI set of vectors in $H$ that spans $H$.

dimension of $H$, denoted by $\dim H$ or $\dim(H)$, is the # vectors in any basis of $H$.

$\dim(\{\bar{0}\}) = 0$ as $\{\bar{0}\}$ is not LI.

$\text{rank}(A) = \dim(\text{Col } A) = $ # pivot columns

$\dim(\text{Nul } A) = $ # free variables

A basis for Col A : pivot columns of A

A basis for Nul A : vectors in parametric vector form of solutions to $A\bar{x} = \bar{0}$.

Rank theorem

For $A \in \mathbb{R}^{m \times n}$,

$$\boxed{\underbrace{\text{rank}(A)}_{\#\text{ pivot columns}} + \underbrace{\dim(\text{Nul } A)}_{\#\text{ free variables}} = \underbrace{n}_{\text{total }\#\text{ columns}}}$$

## Invertible matrix theorem (continued...)

(a) $A \in \mathbb{R}^{n\times n}$ is invertible.

(m) Columns of $A$ form a basis for $\mathbb{R}^n$.

(n) $\text{Col } A = \mathbb{R}^n$.

(o) $\dim \text{Col } A = n$.

(p) $\text{rank } A = n$.

(q) $\text{Nul } A = \{\bar{0}\}$.

(r) $\dim(\text{Nul } A) = 0$.

## Coordinates

We could use any basis to represent a subspace. In any given basis, we can talk about the "coordinates" of any vector in the subspace.

Let $B = \{\bar{b}_1, \dots, \bar{b}_p\}$ be a basis for a subspace $H$. Each $\bar{x} \in H$ can be written as $\bar{x} = c_1\bar{b}_1 + c_2\bar{b}_2 + \dots + c_p\bar{b}_p$ for scalars $c_1, c_2, \dots, c_p$, which are unique to $\bar{x}$. These scalars are called the coordinates of $\bar{x}$ relative to the basis $B$. Stacking these scalars into a vector

$$[\bar{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$ is the $B$-coordinate of $\bar{x}$.

When $B$ is the standard basis, i.e., $\{\bar{e}_1, \dots, \bar{e}_p\}$, the $B$-coordinate of any $\bar{x}$ consists of its own entries.
(So, the $B$-coordinate of $\bar{x}$ is $\bar{x}$ itself here)

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In Exercises 1 and 2, find the vector **x** determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$. Illustrate your answer with a figure, as in the solution of Practice Problem 2.

**1.** $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

We just have to evaluate the linear combination given

$$\bar{x} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

In Exercises 3–6, the vector **x** is in a subspace $H$ with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the $\mathcal{B}$-coordinate vector of **x**.

**5.** $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ 9 \\ -7 \end{bmatrix}$

$\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$. Find $[\bar{x}]_{\mathcal{B}}$. So, find $c_1, c_2$ such that

$$\bar{x} = c_1\bar{b}_1 + c_2\bar{b}_2$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 4 & -7 & 9 \\ -3 & 5 & -7 \end{array}\right] \xrightarrow[R_3+3R_1]{R_2-4R_1} \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right] \xrightarrow[R_1+2R_2]{R_3+R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = 4$, $c_2 = 1$

or $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Why study coordinates?

One often works with a nonstandard basis for a subspace. Hence we want to study how any vector is expressed in such a basis. For instance, we could start with the standard basis for an image, and do a bunch of geometric transformations, e.g., rotate 90° ccw. After that step, we could just work with the nonstandard basis for further analyses.

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In Exercises 17 and 18, mark each statement True or False. Justify each answer. Here $A$ is an $m \times n$ matrix.

**17.** a. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace $H$ and if $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then $c_1, \dots, c_p$ are the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$.

b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

c. The dimension of Col $A$ is the number of pivot columns in $A$.

d. The dimensions of Col $A$ and Nul $A$ add up to the number of columns in $A$.

e. If a set of $p$ vectors spans a $p$-dimensional subspace $H$ of $\mathbb{R}^n$, then these vectors form a basis for $H$.

(a). True. Definition of coordinates.

(b) False. Only if it goes through the origin.

(c) True. Definition.

(d) True. Rank theorem

(e) True. A set of $p$ vectors that spans a $p$-dimensional subspace will be LI. Hence it's a basis.

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**23.** If possible, construct a $3 \times 5$ matrix $A$ such that dim Nul $A = 3$ and dim Col $A = 2$.

$n=5$, dim Col A + dim Nul A $= 2 + 3 = n$ ✓

(so, rank theorem holds)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ works. (has 2 pivot columns).

# Determinants (Section 3.1)

Recall, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, when

determinant of $A$, $\det A = ad - bc \neq 0$.

In general, for $A \in \mathbb{R}^{n\times n}$, $A$ is invertible if and only if $\det A \neq 0$.

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Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}$. When is $A$ invertible?
(what condition should $a, b, c, d$ satisfy in order to make $A$ invertible?)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & b-12 \\ 0 & c-10 & d-15 \end{bmatrix} \quad a-8 \neq 0 \quad \xrightarrow{R_3 - \left(\frac{c-10}{a-8}\right)R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & (b-12) \\ 0 & 0 & (d-15) - \left(\frac{c-10}{a-8}\right)(b-12) \end{bmatrix} \neq 0$$

as we need three pivots

$$(a-8)(d-15) - (c-10)(b-12) \neq 0$$

$$(ad - 15a - 8d + \cancel{120}) - (bc - 12c - 10b + \cancel{120}) \neq 0$$

$$(ad - bc) - (8d - 10b) + (12c - 15a) \neq 0$$

i.e., we need $1(ad-bc)-2(4d-5b)+3(4c-5a) \neq 0$ ← expanding along Row 1

$$\det\left(\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad a \quad b \\ 5 \quad c \quad d \end{bmatrix}\right) = 1\cdot\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - 2\det\left(\begin{bmatrix} 4 & b \\ 5 & d \end{bmatrix}\right) + 3\cdot\det\left(\begin{bmatrix} 4 & c \\ 5 & a \end{bmatrix}\right)$$

We will see that one could expand along any row or any column to compute $\det A$. More on the details in the next lecture.