MATH 220 – Lecture 25 (11/12/2013)

Determinants using cofactor expansion

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation. (Pg 168)

$$\textbf{13.}\quad \begin{array}{c c} & \begin{array}{ccccc} & & 3 & & \end{array} \\ \begin{array}{c} \\ 2 \\ \\ \\ \\ \end{array} & \begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} \end{array} = 2(-1)^{(2+3)} \begin{array}{c c} & \begin{array}{cccc} & 2 & & \end{array} \\ \begin{array}{c} \\ 2 \\ \\ \\ \end{array} & \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} \end{array} = -2\cdot 3(-1)^{(2+2)} \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$$

look for row/column with lots of zeros to expand along!

$$= -6\left(4(-1)^{1+1}\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + 5(-1)^{(2+1)}\begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}\right)$$

$$= -6\left(4\cdot 1 - 5\cdot 1\right) = 6.$$

$$\textbf{12.}\quad \begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix}$$

lower triangular matrix – all entries above the main diagonal are zero.

$$= 4\cdot(-1)^{1+1}\begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} = 4\cdot(-1)\cdot(-1)^{1+1}\begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix} = -4\,(3\times -3)$$

$$= 36$$

Def If all entries above main diagonal are zero, then we have a lower triangular matrix. If all entries below main diagonal are zero, we have an upper triangular matrix.

Theorem 2 If A is a triangular matrix, then

$\det A$ = product of the entries in the main diagonal of A.

Properties of determinants (Section 3.2)

pg 168. In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant. (from 3.1)

$R_1 + kR_2$

22. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

The ERO does not change the determinant here!

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$\det\left(\begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}\right) = (a+\cancel{kc})d - (b+\cancel{kd})c = ad - bc$$

kR_2

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

$$\det\left(\begin{bmatrix} a & b \\ kc & kd \end{bmatrix}\right) = akd - bkc = k(ad - bc)$$

determinant is scaled by k as well

Similarly,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \rightleftarrows R_2} \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\det\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = cb - ad = -(ad - bc)$$

determinant changes sign

Theorem 3 $A, B \in \mathbb{R}^{n \times n}$

1. $A \xrightarrow{R_i + kR_j} B$ $\quad \det B = \det A$

   replacement EROs do not change determinant

2. $A \xrightarrow{R_i \rightleftarrows R_j} B$ $\quad \det B = -\det A.$

   swap changes sign of determinant

3. $A \xrightarrow{kR_i} B$ $\quad \det B = k \cdot \det A$

   scaling scales determinant by same number

25.3

So we could use EROs carefully to evaluate determinants!
avoid scaling EROs

Pg 175 Find the determinants in Exercises 5–10 by row reduction to echelon form.

6. $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{vmatrix} \xrightarrow{R_3+\frac{1}{6}R_2} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 0 & 1 \end{vmatrix} = 1\times -18\times 1 = -18.$

Find the determinants in Exercises 15–20, where (pg 175)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ as the scaling $5R_3$ gives the new matrix

$= 5\cdot 7 = 35.$

Let $A \in \mathbb{R}^{n\times n}$ be row reduced to U in echelon form using r swaps and any number of replacement EROs (and no scaling EROs). Then

$$\det A = \begin{cases} (-1)^r (\text{product of pivot entries in } U) & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases}$$

Notice that we do not need scaling EROs to convert A to echelon form. We might need them to go to reduced echelon form. But echelon form is sufficient here!

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

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25. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

$$A = \begin{bmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{bmatrix} \xrightarrow[R_3+\frac{6}{7}R_1]{R_2+\frac{4}{7}R_1} \begin{bmatrix} 7 & -8 & 7 \\ 0 & 3/7 & 4 \\ 0 & 1/7 & 1 \end{bmatrix} \xrightarrow{R_2-3R_3} \begin{bmatrix} 7 & -8 & 7 \\ 0 & 0 & 1 \\ 0 & 1/7 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \rightleftarrows R_2} \begin{bmatrix} 7 & -8 & 7 \\ 0 & 1/7 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det A = -\left(7 \times \frac{1}{7} \times 1\right) = -1 \neq 0$$

1 swap

So the vectors are LI.

Theorem 4 A is invertible if and only if $\det A \neq 0$.

Theorem 5 $\det(A^T) = \det A$.

Theorem 6 $A, B \in \mathbb{R}^{n\times n}$. $\det(AB) = \det A \cdot \det B$

determinant of a product of two matrices is the product of the individual determinants.

Warning! $\det(A+B) \neq \det A + \det B$ in general.

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29. Compute det B^5, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

$$\det(B^5) = \det(B \cdot B \cdot B \cdot B \cdot B) = (\det B)^5.$$

$$\det B = 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -3 + 1 = -2.$$

$$\det(B^5) = (-2)^5 = -32.$$