MATH 220 – Lecture 3 (08/27/2013)

Recall: If the augmented matrix of a linear system has a row of the form $[0\ 0 \dots 0 \mid * \neq 0]$, then the system is inconsistent.

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$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

Find an equation connecting g, h, k so that this is the augmented matrix of a consistent system.

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$$

need to be zero!

$k+2g+h=0$ for the system to be consistent.

The "simpler" augmented matrices from which we could make conclusions about the existence (and uniqueness) of solutions have a nice, step-like structure. All entries below the "steps" are zero! We will now formalize this concept by introducing the echelon form and the reduced echelon form of a matrix.

Echelon form of a matrix

"steplike"

We need some definitions first.

nonzero row : a row that has at least one nonzero entry

leading entry : the left most nonzero entry of a nonzero row.

Def A matrix is in **row echelon form** if

(1) all nonzero rows are above any zero rows, and

(2) the leading entry of each nonzero row is in a column to the right of the leading entry of the row above.

A consequence of (1) & (2) is that

(3) all entries in a column below a leading entry are zero.

A matrix is in **reduced row echelon form** (RREF) if

(4) each leading entry is 1, and

(5) each leading 1 is the only nonzero entry in its column, i.e., all other entries in the column are zero.

The word "row" is understood in our discussions, and hence we often talk just about echelon form and reduced echelon form.

Examples

$$\begin{bmatrix} 3 & 5 & 0 & 9 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$ is in echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ is in reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$ is not in echelon form, as the zero row (2^{nd} row) is not below all nonzero rows.

Given any matrix, we can do EROs to transform it to echelon form, and further to reduced echelon form. This process is called **row reduction**.
The procedure is called row reduction algorithm.

A matrix is row equivalent to any of its echelon forms.

A matrix can have multiple echelon forms, but its **reduced echelon form is unique**.

Def The leading entries are called pivot elements (or **pivots**). The columns with a pivot are called **pivot columns**.

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Row reduce to echelon form, and to reduced echelon form. Circle pivot entries in both the final and original matrices, and mark the pivot columns.

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix}$$

pivot columns

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix} \xrightarrow{R_2 \rightleftarrows R_3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & -3 & -12 & -18 \\ 0 & 0 & -3 & -6 \end{bmatrix} \xrightarrow[R_3 \times (-1/3)]{R_2 \times (-1/3)}$$

in echelon form

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & -4 & -7 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2-4R_3]{R_1+4R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

in reduced echelon form

As illustrated above there is no harm in combining more than one EROs in one step of row reduction. Of course, one needs to be very careful to specify the particular order in which the multiple EROs in a step are to be executed.