MATH 220 - Lecture 8 (09/12/2013)

Special cases of linear (in)dependence

1. $\{\bar{v}\}$ is LI if $\bar{v} \neq \bar{0}$.

2. $\{\bar{v}_1, \bar{v}_2\}$ is LI if one of them is not a scalar multiple of the other vector.

   if $\bar{v}_1 = c\bar{v}_2$ for scalar c, then $\bar{v}_1 - c\bar{v}_2 = \bar{0}$. So

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -c \end{bmatrix}$ is a nontrivial solution to $\bar{v}_1 x_1 + \bar{v}_2 x_2 = \bar{0}$,

and hence the set of vectors is LD.

3. If $\bar{0}$ is in the set $\{\bar{v}_1, \dots, \bar{v}_n\}$, the set is LD.

   e.g., let $\bar{v}_2 = \bar{0}$. Then

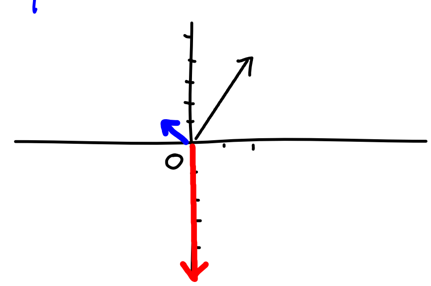
$$0\bar{v}_1 + c\bar{v}_2 + 0\bar{v}_3 + \dots + 0\bar{v}_n = \bar{0} \text{ for any } c \neq 0.$$

Hence $x_1 = 0, x_2 = c, x_3 = x_4 = \dots = x_n = 0$ is a nontrivial solution.

4. $\{\bar{v}_1, \dots, \bar{v}_n\}$, where $\bar{v}_j \in \mathbb{R}^m$, with $n > m$ is LD.

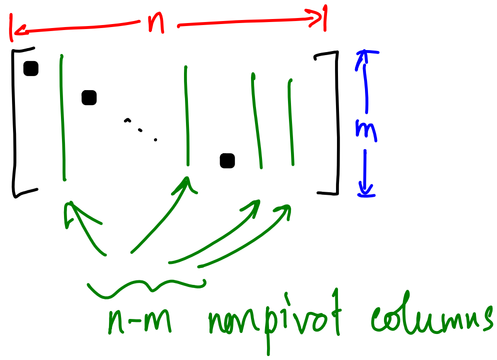
There are more vectors than the number of entries in each vector.

e.g., $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is LD.



Notice that any two vectors out of the three are LI.

Consider $A\bar{x}=\bar{0}$, where $A \in \mathbb{R}^{m\times n}$ with $A = [\bar{v}_1\ \bar{v}_2 \cdots \bar{v}_n]$.



The maximum number of pivots possible is m. So, there are $n-m$ free variables.

Prob 21, pg 61 T/F statements

21. a. The columns of a matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

b. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

c. The columns of any 4×5 matrix are linearly dependent.

d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$.

(a) F. The columns are LI if $A\bar{x}=\bar{0}$ has **only** the trivial solution.
Recall that $A\bar{x}=\bar{0}$ always has the trivial solution.

(b) F. $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ is LD, but $\begin{bmatrix}1\\2\end{bmatrix} \neq c\begin{bmatrix}0\\0\end{bmatrix}$ for any c.
It is only required that one vector in S is a linear combination of the others, not each vector.

(c) T. A set of n vectors each with m entries is LD if $n>m$.

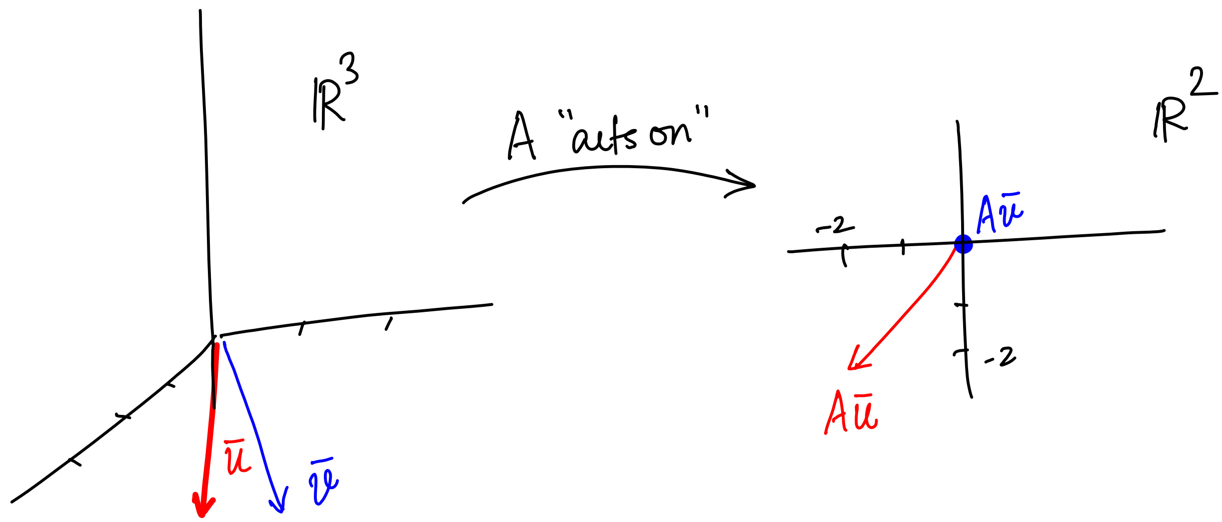
(d) T. $\bar{z}$ can be written as a linear combination of $\bar{x}$ & $\bar{y}$.
$\{\bar{x},\bar{y},\bar{z}\}$ is LD $\Rightarrow$ $a\bar{x}+b\bar{y}+c\bar{z}=0$ for $a,b,c \in \mathbb{R}$, not all being zero. But $c\neq 0$, as $\{\bar{x},\bar{y}\}$ is LI (so $c=0$ would mean $a=b=0$ as well). Hence
$\bar{z} = \left(-\frac{a}{c}\right)\bar{x} + \left(-\frac{b}{c}\right)\bar{y}$, i.e., $\bar{z} \in \text{span}\{\bar{x},\bar{y}\}$.

Linear Transformations (LT) (Section 1.8)

"mappings"

We now talk about another context in which the matrix-vector product $A\bar{x}$ shows up, which is somewhat different from the systems $A\bar{x}=\bar{b}$ that we've been discussing so far.

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}}, \quad \bar{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$



$$A\bar{u} = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\times 3 + -5\times 1 + -7\times 0 \\ -3\times 3 + 7\times 1 + 5\times 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

$$A\bar{v} = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

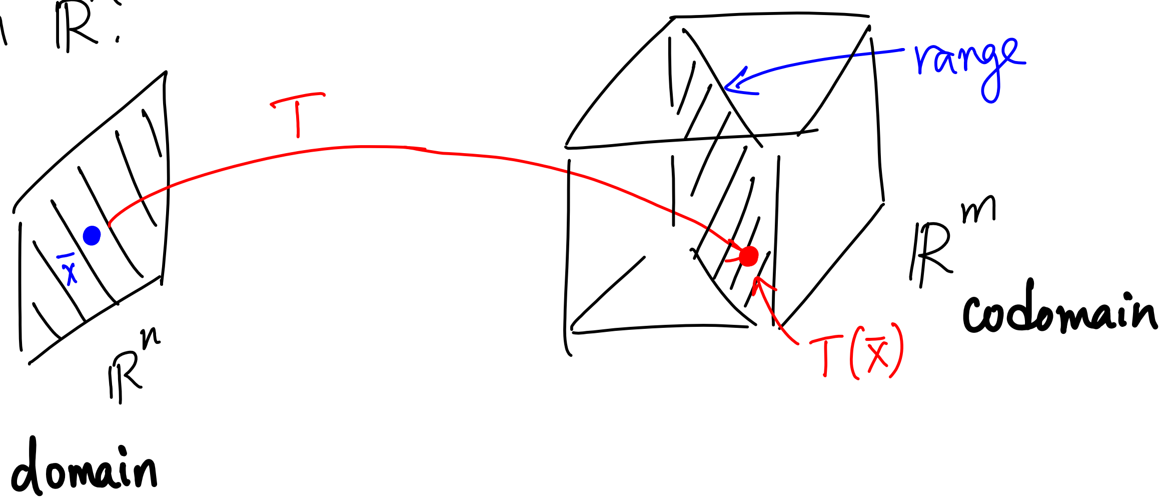
A "acts on" $\bar{u}$ to transform it to $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

In general, A "acts on" $\bar{x} \in \mathbb{R}^3$ to give $A\bar{x} \in \mathbb{R}^2$.

Another notation is to write

$\bar{x} \mapsto A\bar{x}$. This correspondence is a function.

Def A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns every vector $\bar{x}$ in $\mathbb{R}^n$ a vector $T(\bar{x})$ in $\mathbb{R}^m$.



$T(\bar{x})$ is the **image** of $\bar{x}$ under T.

The set of all images is the **range** of T.

We are interested in **matrix transformations**, which are defined as follows.

$T: \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^m}_{\text{codomain}}$ is defined as $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{m \times n}$.

$\bar{x} \mapsto A\bar{x}$ is another notation.

Prob 2, pg 68

$$A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Find images of $\bar{u}$ and $\bar{v}$ under $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\bar{x}) = A\bar{x}$.

$$T(\bar{u}) = A\bar{u} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

$$T(\bar{v}) = A\bar{v} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a \\ \frac{1}{3}b \\ \frac{1}{3}c \end{bmatrix} = \frac{1}{3} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Prob 5, pg 68

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

Find $\bar{x}$ such that $T(\bar{x}) = A\bar{x} = \bar{b}$. Is $\bar{x}$ unique?

We have seen how to solve the system $A\bar{x} = \bar{b}$, and to decide if the system has a unique solution when it is consistent. The same results could be used to answer such questions about linear transformations.

This problem will be finished in the next lecture...