MATH 230 - Lecture 1 (01/11/2011)

My name is Bala Krishnamoorthy (call me Bala).

Research: optimization, algebraic topology, computational biology, and more.

Web page: www.wsu.edu/~kbala

Course web page: www.wsu.edu/~kbala/Math230.html
(see Syllabus and other details here).

Linear Algebra - Motivating example

Butch M. Cougar has 5 hrs (7pm-midnight), and \$48 dollars to spare.

"Math"

Two options:
- get tutoring → \$8/hr
- party → \$16/hr

How many hrs to get tutored, and how many to party?

Let x_1 = # hrs of tutoring
x_2 = # hrs of partying
} variables

Model:

$$\left.\begin{aligned} x_1 + x_2 &= 5 \quad \text{(total hrs)} \\ 8x_1 + 16x_2 &= 48 \quad \text{(total money)} \end{aligned}\right\}$$

System of two linear equations

A generic linear equation : $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

coefficients

right-hand side (rhs)

In general, we talk about a system of m-linear equations in n variables.

The graph of a linear equation is a straight line.

Here are some equations that are NOT linear:

$$2x_1x_2 + 3x_3 = -4$$

$$-\sqrt{x_2} + 3x_3^5 = 0$$

non-linear terms.

The coefficients (a_i) and rhs (b) can be real or complex numbers.
In Math 230, we will deal with only real numbers.

A **solution** is a set of values for $x_1, x_2, \ldots$ for which each equation in the system is true (or holds).
The **solution set** is the set of all solutions.

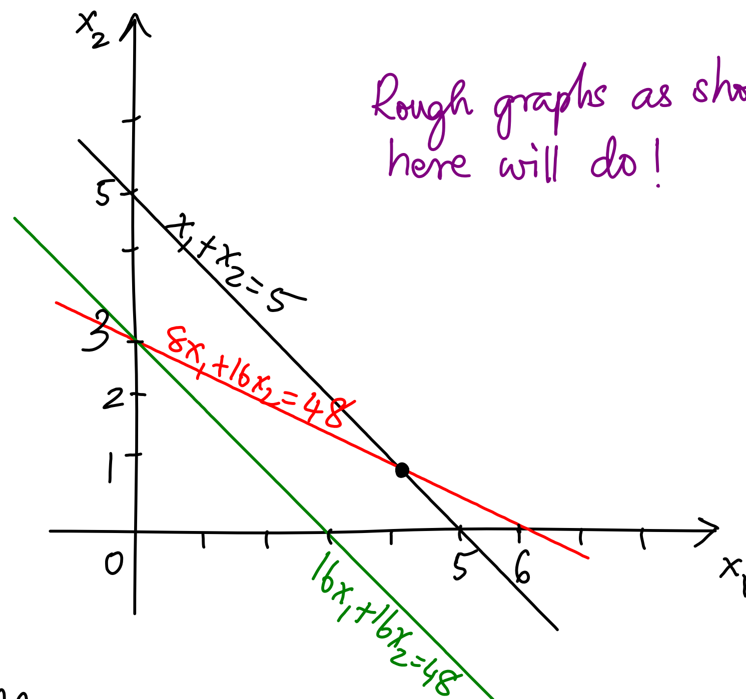
$$\left.\begin{aligned} x_1 + x_2 &= 5 \quad \text{--- (1)} \\ 8x_1 + 16x_2 &= 48 \quad \text{--- (2)} \end{aligned}\right\} \text{A solution is given by} \quad \left.\begin{aligned} x_1 &= 4 \\ x_2 &= 1 \end{aligned}\right\}$$

find solution by plotting the graphs of the two equations.

Find two points per line:

$x_1 + x_2 = 5$ } $(5,0)$ and $(0,5)$ are points on this line

$8x_1 + 16x_2 = 48$ } $(6,0)$ & $(0,3)$ are points



Rough graphs as shown here will do!

How many solutions?

The 2 lines intersect at exactly one point. So, the solution set is $\{(4,1)\}$, i.e., it is a singleton set.

There are two other extreme cases

* The 2 lines do not intersect $\Rightarrow$ "implies" system has no solution, OR the system is **inconsistent**.

  e.g., fees for tutoring is \$16/hr now. The system is

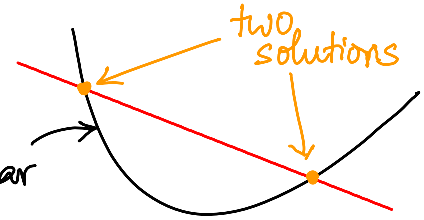
  $$\left.\begin{aligned} x_1 + x_2 &= 5 \\ 16x_1 + 16x_2 &= 48 \end{aligned}\right\}$$ The two lines are parallel.

* The 2 lines intersect at ALL pts $\Rightarrow$ the system has infinitely many solutions.

  e.g., tutoring is \$16/hr, and Butch has only 3 hrs to spare:

  $$\left.\begin{aligned} x_1 + x_2 &= 3 \\ 16x_1 + 16x_2 &= 48 \end{aligned}\right\}$$ The two lines are the same!

Note: We never get, say, 2 solutions with lines! We could get 2 points of intersection if we had curve(s) instead of lines.



In summary, a system of linear equations can have

1. no solution; $\longrightarrow$ inconsistent system
2. exactly one solution; OR
3. infinitely many solutions

} consistent system

The idea of plotting lines will not work in higher dimensions.

In general, we use elimination to solve a system of linear equations.

$\hookrightarrow$ From the original system, obtain an "equivalent" system that is "simpler".

both systems have the same solution set.

$$\underbrace{\left.\begin{aligned} x_1 + x_2 &= 5 \quad \text{—(1)} \\ 8x_1 + 16x_2 &= 48 \quad \text{—(2)} \end{aligned}\right\}}_{\text{original system}} \longrightarrow \left.\begin{aligned} x_1 + 0x_2 &= 4 \\ 0x_1 + x_2 &= 1 \end{aligned}\right\} \text{simpler equivalent system.}$$

How? Eliminate x_1 from equations (2), (3), ..., eliminate x_2 from equations (1), (3), ..., etc.

$$\begin{aligned} x_1 + x_2 &= 5 \quad \text{---(1)} \\ 8x_1 + 16x_2 &= 48 \quad \text{---(2)} \end{aligned} \xrightarrow{-8\times(1)+(2)} \begin{aligned} x_1 + x_2 &= 5 \quad \text{---}(1') \\ 8x_2 &= 8 \quad \text{---}(2') \end{aligned} \quad (2')\times\frac{1}{8}$$

$$\begin{aligned} x_1 + x_2 &= 5 \quad \text{---}(1'') \\ x_2 &= 1 \quad \text{---}(2'') \end{aligned} \xrightarrow{(1'')-(2'')} \begin{aligned} x_1 &= 4 \\ x_2 &= 1 \end{aligned}$$

This procedure is called Gaussian elimination. The operations are Elementary row operations (**EROs**).

they do not change the solution set.

There are **three** types of EROs.

1. (**Replacement**): Replace an equation with the sum of itself and a multiple of another equation.
2. (**Interchange**): Swap two equations (or rows). just change the order in which equations are written
3. (**Scaling**): Multiply an equation (row) by a **non-zero** number.

If you multiply by zero, you remove that equation!

More compact representation! Avoid writing $x_1, x_2, \ldots$ each time.

$$\begin{aligned} x_1 + x_2 &= 5 \\ 8x_1 + 16x_2 &= 48 \end{aligned} \qquad \begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix}$$ is the matrix of coefficients.

A **matrix** is a rectangular array of numbers. It has rows and columns.

The *size* of a matrix: (# rows) $\times$ (# columns)

said as "by"

$\begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix}$ is a 2×2 matrix.

We include the rhs numbers along with the matrix of coefficients:

$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 16 & 48 \end{array}\right] \longrightarrow$ the **augmented matrix** of the system of linear equations

just a separator line; separates the rhs from coefficients.

$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 16 & 48 \end{array}\right]$ is a 2×3 matrix.

We can do EROs directly on the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 16 & 48 \end{array}\right] \xrightarrow{R_2-8R_1} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 8 & 8 \end{array}\right] \xrightarrow{R_2\times(\frac{1}{8})} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

compact notation for EROs.

MATH 230 – Lecture 2 (01/13/2011)

Elementary row operations (EROs) can be applied to ANY matrix. We do so on augmented matrices of systems of linear equations to find their solution(s).

Prob 14, pg 11

Solve the given system.

$$\begin{aligned} x_1 - 3x_2 \quad\;\; &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \rightleftharpoons R_3} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

$$\xrightarrow[R_3+2R_2]{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow{R_3 \times \left(\frac{1}{7}\right)} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-3R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow$$

This is the augmented matrix of a simpler system:

$$\begin{aligned} x_1 \quad\quad &= 2 \\ x_2 \quad &= -1 \\ x_3 &= 1 \end{aligned}$$

Note that this system belongs to Case 2, i.e., it has a unique solution.

Two matrices are **row equivalent** if we get one from the other using EROs. If we get the second matrix from the first using EROs, we can get the first from the second using another set of EROs.

EROs are reversible.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{R_1+3R_3}{\overset{R_1-3R_3}{\rightleftarrows}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Two systems of linear equations are **equivalent** if they have the same solution set. Note that in this case, their augmented matrices are row equivalent.

<u>Prob 11, pg 11</u>

Solve the system:

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{array}\right] \xrightarrow{R_1 \rightleftarrows R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{array}\right] \xrightarrow{R_3-3R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

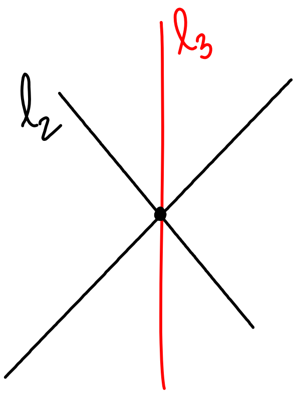
This row corresponds to $0x_1 + 0x_2 + 0x_3 = 2$, i.e, $0 = 2$!

Hence the original system is inconsistent, i.e., it has no solution.

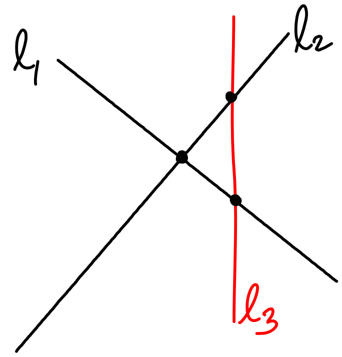
In general, if you get **[0 0 ... 0|*]** for *** non-zero**, the original system is **inconsistent.**

What if we get [0 0 .. 0 | 0] ?

This is o.k., i.e., the system is consistent, as we just have $0=0$, which is true. Such a row means that one equation in the original system is "redundant".



This is the graph of of 3 equations in two variables. The system is consistent, as ALL 3 lines meet at the same point. We get this point by using lines (l_1, l_2), (l_1, l_3), or (l_2, l_3) alone. Hence, one equation is redundant.



This is the graph of a system of 3 equations in 2 variables that is inconsistent. Notice that all three lines do not meet at the same point. But every pair of lines gives a consistent system.

Existence and uniqueness questions

1. Is the system consistent?
2. If there is a solution, is it the unique solution?

We'll see these questions again and again and again ...

Prob 20, pg 12

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$ For what values of h is this the augmented matrix of a consistent system?

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$$

can be any value!

We cannot get $[0 \ 0 | *]$ for $*$ nonzero. Hence the system is consistent for all values of h, ie, $h \in \mathbb{R}$

"in" or "element of"!

set of all real numbers

Prob 25, pg 12

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

Find a relation between g, h, k such that this is the augmented matrix of a consistent system.

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$$

We need $k+2g+h=0$ for the system to be consistent. Else, we get $[0 \ 0 \ 0 | *]$ for $*$ nonzero.

TRUE/FALSE (Prob 23, pg 12)

Write justifications as illustrated here.

(b) A 5x6 matrix has 6 rows.

FALSE. It has 5 rows and 6 columns.

(*) A system of 3 linear equations in 3 variables can have 3 solutions.

FALSE. If it has more than one solution, it has infinitely many solutions, not 3.

Echelon form of a matrix (Section 1.2)

Def: **Non-zero row**: a row that has at least one nonzero entry

leading entry: The leftmost non-zero entry in a non-zero row.

A matrix is in **row echelon form**, or simply, in **echelon form**, if

(1) all nonzero rows are above any zero rows; and

(2) the leading entry of each row is in a column that is to the right of the leading entry of the row above it.

(1) & (2) imply that

(3) all entries in a column below a leading entry are zero.

A matrix is in **reduced row echelon form (RREF)** if it is in echelon form, and

(4) the leading entry in each non-zero row is 1, and

(5) the leading 1 is the only nonzero entry in its column. (all other entries are zero).

e.g., $\begin{bmatrix} 3 & 5 & 0 & 9 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ is in <u>echelon</u> form

"step like"

$\begin{bmatrix} 1 & 0 & \cancel{0}\,1 & 0 & 2 \\ 0 & 1 & \cancel{0}\,5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form.

still in reduced echelon form.

General notation for echelon forms

$\blacksquare \rightarrow$ non-zero, $\quad * \rightarrow$ zero or non-zero

$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$ is in reduced echelon form.

What are **all** possible echelon forms of a 3x3 matrix?

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any matrix can be converted to an echelon form, and to reduced echelon form using EROs.

A matrix can have several echelon forms, but its **reduced echelon form is unique.**

To solve a system of linear equations, we take its augmented matrix to echelon form, and to reduced echelon form if needed (i.e., if it is consistent).

MATH 230 – Lecture 3 (01/18/2011)

Echelon form and reduced echelon form

Row reduction is the procedure of using EROs to reduce **any** matrix into echelon form, and then to reduced echelon form.

Def :

pivot → the leading entry in a nonzero row

pivot position → position of a leading entry (same as pivot)

pivot column → a column that contains a pivot.

Prob 4, pg 25 Row reduce the given matrix to reduced echelon form. Circle echelon form, and then to reduced echelon form. Circle the pivots, list the pivot columns.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

is in echelon form

$$\xrightarrow[R_3\times\left(-\frac{1}{10}\right)]{R_2\times\left(-\frac{1}{4}\right)} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-3R_3]{R_1+2R_3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot columns

is in reduced echelon form

Columns 1, 2, and 4 are pivot columns.

Solution of linear systems: Use row reduction to convert the augmented matrix to echelon form, and if found consistent, reduce further to reduced echelon form to write the solutions down.

<u>Prob 12, pg 25</u> The augmented matrix of a system of linear equations is given. Find all solutions.

pivots are circled

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{rhs} \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right] \xrightarrow{R_3+4R_2} \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot columns 1 & 3 correspond to variables x_1 and x_3.

<u>Def.</u> Variables corresponding to pivot columns are called **basic** variables. Variables that are not basic are **free** variables (or non-basic variables).

Here, x_1, x_3 are basic and x_2, x_4 are free variables.

We can choose the values of free variables arbitrarily. The basic variables can be written in terms of the free variables.

A system of linear equations has **infinitely many solutions** if and only if it **has free variables**

Back to prob 12, pg 25

The reduced echelon form is $\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

$$\left.\begin{aligned} x_1 - 7x_2 + 6x_4 &= 5 \\ x_3 - 2x_4 &= -3 \end{aligned}\right\} \quad \boxed{\begin{aligned} x_1 &= 5 + 7x_2 - 6x_4 \\ x_3 &= -3 + 2x_4 \\ &x_2, x_4 \text{ are free} \end{aligned}}$$

general solution to the original system.

Summary of linear systems

1. If the echelon form of the augmented matrix has a row $[0\ 0 \cdots 0 \,|\, *]$ for $*$ non-zero, the system is inconsistent.

   Else

2. If there are no free variables, the system has a unique solution.

3. If there are free variables, the system has infinitely many solutions.

Prob 15(a) pg 25

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$$

is the augmented matrix in echelon form. Is the system consistent?

YES, as there is no row of the form $[0\ 0\ 0\ *]$ for $*$ nonzero!

Is the solution unique?

YES. As there are no free variables.

Prob 25, pg 26 The coefficient matrix has a pivot in each row. Why is the system consistent?

There is at least one non-zero entry in each row of the coefficient matrix. Hence we cannot have an inconsistent row $([0\ 0 \ldots 0 | \blacksquare])$ in the augmented matrix.

Does this system have a unique solution?

Don't know.

There could be free variables even when there is a pivot in each row, in which case, the solution is not unique.

Prob 19, pg 26

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

Choose h, k such that system is
(a) inconsistent;
(b) has unique solution; and
(c) has many solutions.

ALSO, in cases (b) and (c), write all solutions down.

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right] \xrightarrow{R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right]$$

(a) We need $8-4h=0$ and $k-8 \neq 0$, so that we get $[0\ 0 \mid \blacksquare]$. So, $h=2$ and $k \neq 8$.

can also write as $k \in \mathbb{R}/\{8\}$

set of all reals without 8.

(b) $8-4h \neq 0$ so that there are no free variables.

$h \neq 2$, k is any real number. $h \in \mathbb{R}/\{2\}$, $k \in \mathbb{R}$.

(c) Need $8-4h=0$ and $k-8=0$ so that x_2 is a free variable in a consistent system.

i.e., $h=2$, $k=8$.

Now, let's find the solution(s) in (b) and (c).

(b) We have $8-4h \neq 0$.

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right] \xrightarrow{R_2 \times \frac{1}{(8-4h)}} \left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 1 & \frac{k-8}{8-4h} \end{array}\right] \xrightarrow{R_1 - hR_2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2-h\left(\frac{k-8}{8-4h}\right) \\ 0 & 1 & \frac{k-8}{8-4h} \end{array}\right]$$

$$x_1 = 2 - h\left(\frac{k-8}{8-4h}\right) = \frac{2(8-\cancel{4h}) - hk + \cancel{8h}}{4(2-h)}$$

$$= \frac{16-hk}{4(2-h)}$$

$$x_2 = \frac{k-8}{8-4h} = \frac{k-8}{4(2-h)}$$

The unique solution is $x_1 = \frac{16-hk}{4(2-h)}$, $x_2 = \frac{k-8}{4(2-h)}$.

(c) $h=2$, $k=8$ gives

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free, x_1 is basic

$$\left.\begin{array}{l} x_1 + 2x_2 = 2 \\ x_2 \text{ free} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = 2 - 2x_2 \\ x_2 \text{ free} \end{array}\right\}$$

general solution.

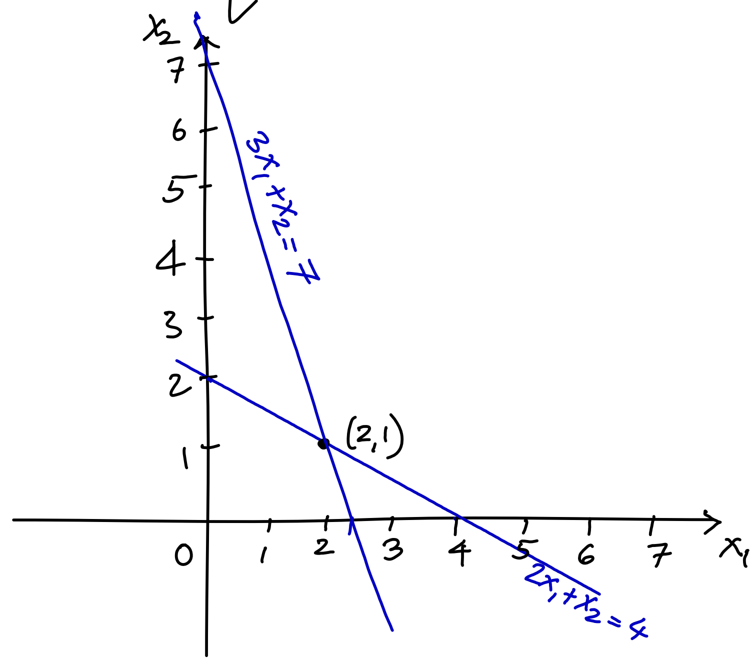
Vector Equations (Section 1.3)

$3x_1 + x_2 = 7$
$x_1 + 2x_2 = 4$

$$\left[\begin{array}{cc|c}3&1&7\\1&2&4\end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c}0&-5&-5\\1&2&4\end{array}\right] \xrightarrow[R_1\rightleftarrows R_2]{R_1\times\left(-\frac{1}{5}\right)}$$

$$\left[\begin{array}{cc|c}1&2&4\\0&1&1\end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c}1&0&2\\0&1&1\end{array}\right] \quad \begin{cases}x_1=2\\x_2=1\end{cases} \text{ unique solution.}$$

This is the "row picture" of the system. We'll now look at the "column picture".



$\left[\begin{array}{cc|c}3&1&7\\1&2&4\end{array}\right]$ corresponds to

$$\left.\begin{bmatrix}3\\1\end{bmatrix}x_1 + \begin{bmatrix}1\\2\end{bmatrix}x_2 = \begin{bmatrix}7\\4\end{bmatrix}\right\} \text{ is a vector equation}$$

vectors.

A vector (or a column vector, by default) is a column of entries. An n-vector is a column with n entries.

"bar": indicates a vector $\bar{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$. $\bar{x}$ here is also a $1\times n$ matrix.

MATH 230 – Lecture 4 (01/20/2011)

Vector equations

$$\left.\begin{aligned} 3x_1 + x_2 &= 7 \\ x_1 + 2x_2 &= 4 \end{aligned}\right\} \quad \begin{bmatrix}3\\1\end{bmatrix} x_1 + \begin{bmatrix}1\\2\end{bmatrix} x_2 = \begin{bmatrix}7\\4\end{bmatrix}$$

$x_1 = 2, x_2 = 1$ is the unique solution

Notation: The set of all n-vectors is denoted by $\mathbb{R}^n$.

e.g., $\bar{x} = \begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \in \mathbb{R}^3$ when $\alpha, \beta, \gamma \in \mathbb{R}$.

$\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}7\\4\end{bmatrix}$ are vectors in $\mathbb{R}^2$.

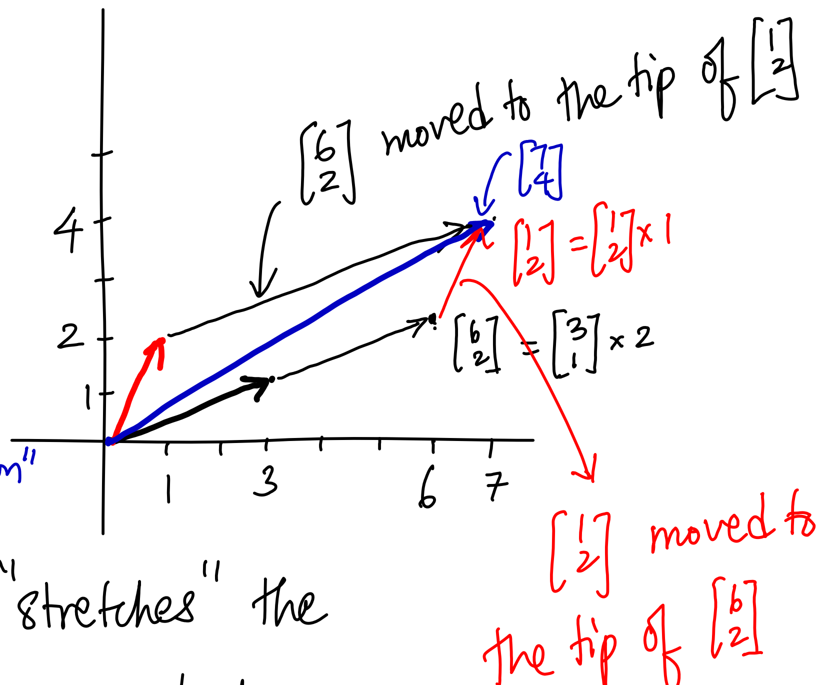
$c\bar{x}$ for $c \in \mathbb{R}$, $\bar{x} \in \mathbb{R}^n$ is vector-scalar multiplication.

also called "scalar multiplication"

If $\bar{x} = \begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}$, then

$$c\bar{x} = \begin{bmatrix}cx_1\\cx_2\\ \vdots\\cx_n\end{bmatrix}.$$

Also, $c\bar{x}$, "stretches" the vector $\bar{x}$ by the factor c.



In words, can you find multipliers x_1 and x_2, which when used to scale the vectors $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$, and added, gives $\begin{bmatrix}7\\4\end{bmatrix}$?

Note: $\bar{u} = [2\ 3\ 5]$ is a row-vector. It is **NOT** the same as $\bar{v} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, which is a column vector.

By default, we assume vectors are column vectors.

Adding scalar multiples of vectors is called "taking a linear combination" of the vectors.

Def: For vectors $\bar{v}_1, \bar{v}_2, \dots \bar{v}_n \in \mathbb{R}^m$, the vector

$$\bar{v} = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_n\bar{v}_n \quad \text{for } c_i \in \mathbb{R} \text{ for all } i, \text{ is}$$

a **linear combination** of $\bar{v}_1, \dots, \bar{v}_n$.

Note: We can add vectors $\bar{u}$ and $\bar{v}$ only if they have the same # entries.

So, $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ is not defined.

The c_i's in the linear combination $\bar{v} = c_1\bar{v}_1 + \dots + c_n\bar{v}_n$ are called **weights** of the linear combination.

e.g., $-3\begin{bmatrix}3\\1\end{bmatrix}+4\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-3\times3\\-3\times1\end{bmatrix}+\begin{bmatrix}4\times1\\4\times2\end{bmatrix}=\begin{bmatrix}-9+4\\-3+8\end{bmatrix}=\begin{bmatrix}-5\\5\end{bmatrix}$

So, $\begin{bmatrix}-5\\5\end{bmatrix}$ is a linear combination of $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ with weights $c_1=-3$ and $c_2=4$.

Note that $\left\{\begin{aligned}3x_1+x_2&=-5\\x_1+2x_2&=5\end{aligned}\right\}$ has the solution $x_1=-3,\ x_2=4$.

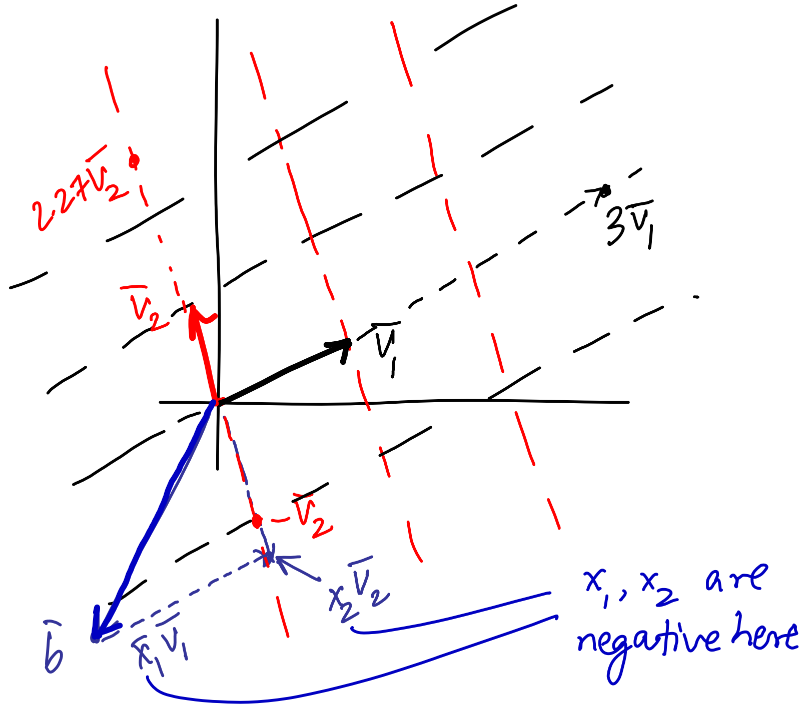
In general, if $\bar{b}=c_1\bar{a}_1+c_2\bar{a}_2+\cdots+c_n\bar{a}_n$, then the system of linear equations whose augmented matrix is $\left[\bar{a}_1\ \bar{a}_2\cdots\bar{a}_n\,\middle|\,\bar{b}\right]$ has a solution $x_1=c_1, x_2=c_2, \ldots, x_n=c_n$.

Equivalently, to answer the question "Is the system with augmented matrix $[\bar{a}_1\ \bar{a}_2\cdots\bar{a}_n\,|\,\bar{b}]$ consistent?" we can answer the question "**Can $\bar{b}$ be written as a linear combination of $\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n$?**"

Def: The set of all linear combinations of $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n \in \mathbb{R}^m$ is called the **span** of $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$, and writted span$\{\bar{v}_1, \cdots, \bar{v}_n\}$.

We can write any vector in $\mathbb{R}^2$ as a linear combination of $\begin{bmatrix}3\\1\end{bmatrix}$ & $\begin{bmatrix}1\\2\end{bmatrix}$.

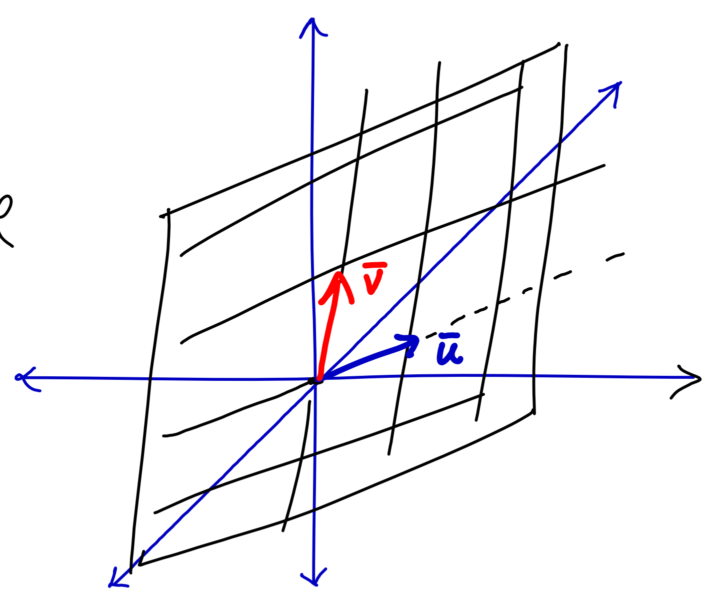
$$\text{span}\left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\} = \mathbb{R}^2.$$



Note that the zero vector, or origin, is in every span.

$$\bar{0} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} = 0\cdot\bar{v}_1 + 0\cdot\bar{v}_2 + \cdots + 0\cdot\bar{v}_n \quad \text{(take all weights as zero).}$$

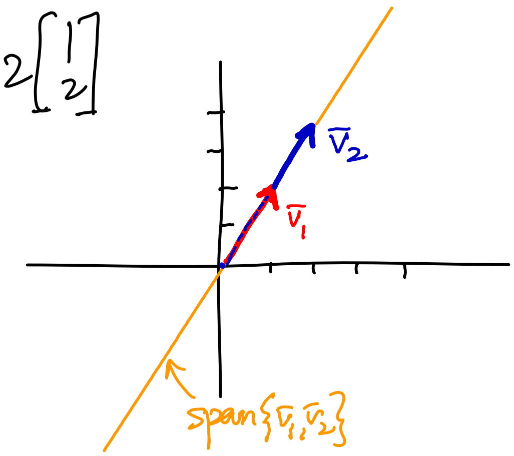
In $\mathbb{R}^3$, the span of the two vectors $\bar{u}, \bar{v}$ shown here is a <u>plane</u> passing through the origin.



But $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$ is just a line in $\mathbb{R}^2$ through the origin, as

$$\bar{v} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\4\end{bmatrix} = c_3\begin{bmatrix}1\\2\end{bmatrix}, \text{ since } \begin{bmatrix}2\\4\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix}$$

$[c_1 + 2c_2]$



Prob 18, pg 38

$\bar{v}_1 = \begin{bmatrix}1\\0\\-2\end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix}-3\\1\\8\end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix}h\\-5\\-3\end{bmatrix}$. For what values of h is $\bar{v}_3$ in the plane generated by $\bar{v}_1$ and $\bar{v}_2$?

refers to span$\{\bar{v}_1, \bar{v}_2\}$.

Reword: For what h is $\bar{v}_3$ in span$\{\bar{v}_1, \bar{v}_2\}$?

Equivalently for what h is $\left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\-2 & 8 & -3\end{array}\right]$ the augmented matrix of a consistent system?

$$\left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\-2 & 8 & -3\end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\0 & 2 & -3+2h\end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\0 & 0 & 7+2h\end{array}\right]$$

Need $7+2h=0$, to avoid $[0\ 0 \mid \blacksquare]$ row.

So, $h = \frac{-7}{2}$.

Properties of $\mathbb{R}^n$ (n-vectors)

$\bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^n$, $\underbrace{c, d}_{\text{scalars}} \in \mathbb{R}$, $\bar{0}$ is the zero vector

(i) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$

(ii) $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$

(iii) $\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}$

(iv) $\bar{u} + (-\bar{u}) = \bar{0}$

(v) $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$

(vi) $(c+d)\bar{u} = c\bar{u} + d\bar{u}$

(vii) $c(d\bar{u}) = (cd)\bar{u} = (c\bar{u})d$

These are properties of addition and multiplication for scalars (i.e., real numbers) extended to the case of vectors.

TRUE/FALSE (Prob 23, pg 38)

(c) An example of linear combination of $\bar{v}_1$ and $\bar{v}_2$ is $\frac{1}{2}\bar{v}_1$.

TRUE. $\frac{1}{2}\bar{v}_1 = \left(\frac{1}{2}\right)\bar{v}_1 + 0\,\bar{v}_2$ is a linear combination of $\bar{v}_1$ and $\bar{v}_2$ with weights $\frac{1}{2}, 0$.

(e) span$\{\bar{u}, \bar{v}\}$ is always a plane through origin.

FALSE. If $\bar{u}$ and $\bar{v}$ are scalar multiples of each other, we get a line, not a plane – see page 4-5 above.

Introduction to MATLAB on http://my.math.wsu.edu.

rref(A) gives the reduced row echelon form of matrix A.

MATH 230 - Lecture 5 (01/25/2011)

The matrix equation $A\bar{x}=\bar{b}$ (Section 1.4)

For the system

$$\left\{\begin{array}{l} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{array}\right\}, \quad \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right] \longrightarrow \text{augmented matrix, and}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \longrightarrow \text{vector equation}$$

The matrix equation is $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

In general, the matrix equation is $A\bar{x}=\bar{b}$, where A is an $m\times n$ matrix with $A=[\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$ corresponding to the system whose augmented matrix is $[A\,|\,\bar{b}]$.

of m equations in n variables

$A\bar{x}$ is a **matrix-vector product**. In general, if $A=[\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$, where $\bar{a}_j \in \mathbb{R}^m$, and $\bar{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

then
$$A\bar{x} = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n}_{\text{linear combination of the } \bar{a}_j\text{'s with weights } x_j\text{'s}}.$$

This is the linear combination of $\bar{a}_1, \dots, \bar{a}_n$ with weights $x_1, x_2, \dots, x_n$. Thus we get the result that $A\bar{x} = \bar{b}$ has a solution if and only if $\bar{b}$ can be written as a linear combination of $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$. Equivalently, if and only if $\bar{b} \in \text{span}\{\bar{a}_1, \dots, \bar{a}_n\}$.

Why study just A?

Motivation: We can answer several questions about the system $A\bar{x} = \bar{b}$ just by doing EROs on A.

Note: The matrix-vector product $A\bar{x}$ is defined only when A is $m \times n$ and $\bar{x}$ is an n-vector, i.e., when (# columns in A) = (# entries in $\bar{x}$).

e.g., $\cancel{\begin{bmatrix} -4 & 2 \\ 0 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}}$ is not defined.

$\begin{bmatrix} -4 & 2 \\ 0 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is defined, and is given by

$$\begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} \times 3 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \times (-2) = \begin{bmatrix} -4 \times 3 + 2 \times -2 \\ 0 \times 3 + 1 \times -2 \\ 6 \times 3 + 3 \times -2 \end{bmatrix} = \begin{bmatrix} -16 \\ -2 \\ 12 \end{bmatrix}$$

Prob 14, pg 48 Let $\bar{u}=\begin{bmatrix}2\\-3\\2\end{bmatrix}$, $A=\begin{bmatrix}5&8&7\\0&1&-1\\1&3&0\end{bmatrix}$.

Is $\bar{u}$ in the subset of $\mathbb{R}^3$ spanned by columns of A?

Reword: Is $\bar{u}$ in span$\{\bar{a}_1,\bar{a}_2,\bar{a}_3\}$, where $A=[\bar{a}_1\ \bar{a}_2\ \bar{a}_3]$?

Or, does the system $A\bar{x}=\bar{u}$ have a solution?

$$\left[\begin{array}{ccc|c}5&8&7&2\\0&1&-1&-3\\1&3&0&2\end{array}\right]\xrightarrow{R_1\rightleftarrows R_3}\begin{bmatrix}1&3&0&2\\0&1&-1&-3\\5&8&7&2\end{bmatrix}\xrightarrow{R_3-5R_1}\begin{bmatrix}1&3&0&2\\0&1&-1&-3\\0&-7&7&-8\end{bmatrix}$$

$$\xrightarrow{R_3+7R_2}\begin{bmatrix}1&3&0&2\\0&1&-1&-3\\0&0&0&-29\end{bmatrix}$$

The system is inconsistent.

So, $\bar{u}\notin$ span$\{\bar{a}_1,\bar{a}_2,\bar{a}_3\}$ for $A=[\bar{a}_1\ \bar{a}_2\ \bar{a}_3]$.

"not element of", or "not in the set"!

We now consider a problem where a particular right-hand side (rhs) or target vector is not given, but questions are asked based only on the coefficient matrix. To demonstrate, we will first solve the problem for a generic rhs vector $\bar{b}$, and then demonstrate how to answer the question without using any $\bar{b}$ vector.

Prob 18, pg 48

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}.$$

Do the columns of B span $\mathbb{R}^4$?

Reword: Can you write any vector $\bar{b} \in \mathbb{R}^4$ as a linear combination of columns of B?

Let $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. Is $\bar{b} \in \text{span}\{\text{columns of } B\}$?

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 2 & b_1 \\ 0 & 1 & 1 & -5 & b_2 \\ 1 & 2 & -3 & 7 & b_3 \\ -2 & -8 & 2 & -1 & b_4 \end{array}\right] \xrightarrow[R_4+2R_1]{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 2 & b_1 \\ 0 & 1 & 1 & -5 & b_2 \\ 0 & -1 & -1 & 5 & b_3-b_1 \\ 0 & -2 & -2 & 3 & b_4+2b_1 \end{array}\right] \xrightarrow[R_4+2R_2]{R_3+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 2 & b_1 \\ 0 & 1 & 1 & -5 & b_2 \\ 0 & 0 & 0 & 0 & b_3-b_1+b_2 \\ 0 & 0 & 0 & -7 & b_4+2b_1+2b_2 \end{array}\right]$$

Need $b_3 - b_1 + b_2 = 0$ for the system to be consistent.

So, not every $\bar{b} \in \mathbb{R}^4$ belongs to span{columns of B}.

The set of all vectors in span{columns of B} is the vectors $\bar{b}$ for which $b_3 - b_1 + b_2 = 0$, i.e., $b_1 = b_2 + b_3$.

Result In general, for $A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$ with $\bar{a}_j \in \mathbb{R}^m$, $\operatorname{span}\{\bar{a}_1, \ldots, \bar{a}_n\} = \mathbb{R}^m$ if and only if (echelon form of) A has a **pivot in every row**.

Prob 22, pg 48 $\bar{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$. Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span $\mathbb{R}^3$?

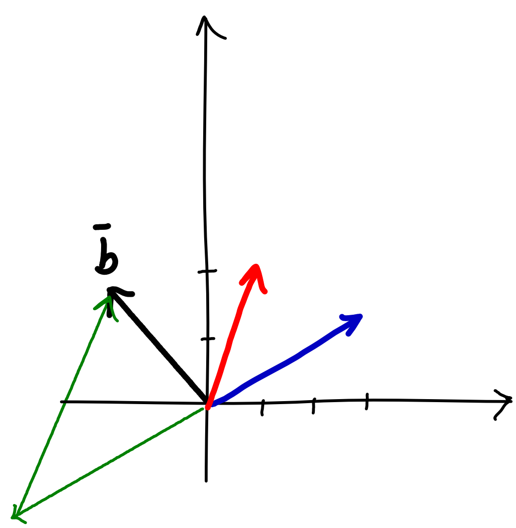
$$[\bar{v}_1\ \bar{v}_2\ \bar{v}_3] = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \xrightarrow{R_3 \rightleftharpoons R_1} \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

There is a pivot in every row. Hence $\operatorname{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \mathbb{R}^3$.

Similarly, $\operatorname{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \mathbb{R}^2$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 0 & -5 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 \rightleftharpoons R_2} \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$$

pivot in every row



Can write any vector $\bar{b} \in \mathbb{R}^2$ as a linear combination of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Note: The echelon form of A can have a pivot in every row with or without having non-pivot columns. Hence we cannot answer questions about uniqueness of solution(s) for every $\bar{b}$ just based on the fact that it has a pivot in every row.

Prob 30, pg 49 Construct a 3x3 **nonzero** matrix, not in echelon form whose columns do not span $\mathbb{R}^3$. Justify your answer.

every entry is $\neq 0$.

The matrix should not have a pivot in every row.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 3 & 4 \end{bmatrix}$$

cannot have a pivot. To make sure, we go to the echelon form here.

$$\xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \rightleftarrows R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

R_1 and R_2 are identical here!

Alternatively, how to find a non-zero matrix whose columns do span $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns here do span $\mathbb{R}^3$, as there is a pivot in every row. Now, use EROs to convert the zero entries to non-zero.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Properties of matrix-vector multiplication

$A \in \underbrace{\mathbb{R}^{m\times n}}_{}$, $\bar{u}, \bar{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$.

set of all $m\times n$ matrices with real entries

(a) $A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$.

(b) $A(c\bar{u}) = c(A\bar{u})$.

<u>Proof for (a)</u> Let $A = [\bar{a}_1 \ \bar{a}_2 \cdots \bar{a}_n]$, where $\bar{a}_j \in \mathbb{R}^m$,

$m\times n$ matrix

and $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

We assume vector-scalar multiplication and vector addition properties here.

$$\bar{u}+\bar{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}.$$

$$A(\bar{u}+\bar{v})=\begin{bmatrix}\bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n\end{bmatrix}\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}=\bar{a}_1(u_1+v_1)+\bar{a}_2(u_2+v_2)+\cdots+\bar{a}_n(u_n+v_n)$$

$$=(\bar{a}_1u_1+\bar{a}_1v_1)+(\bar{a}_2u_2+\bar{a}_2v_2)+\cdots+(\bar{a}_nu_n+\bar{a}_nv_n)$$

as $(c+d)\bar{u}=c\bar{u}+d\bar{w}$ for $c,d\in\mathbb{R}$, $\bar{w}\in\mathbb{R}^n$

$$=\underbrace{(\bar{a}_1u_1+\bar{a}_2u_2+\cdots+\bar{a}_nu_n)+(\bar{a}_1v_1+\bar{a}_2v_2+\cdots+\bar{a}_nv_n)}$$

as $\bar{w}+\bar{v}=\bar{v}+\bar{w}$ for $\bar{u},\bar{v}\in\mathbb{R}^n$

$$=\begin{bmatrix}\bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n\end{bmatrix}\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}\bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n\end{bmatrix}\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$$

$$=A\bar{w}+A\bar{v}.$$

MATH 230 - Lecture 6 (01/27/2011)

Notation for replacement EROs

$$\begin{bmatrix} & \end{bmatrix} \xrightarrow[3R_2+R_1]{R_1+3R_2}$$

R_1+3R_2 → Replace **Row 1** by the sum of itself and $3\times$(Row 2).

$3R_2+R_1$ → Scale **Row 2** by 3 and then replace it by the sum of the scaled version and Row 1.

These two are NOT the same!

Of course, they are both valid EROs.

Another somewhat confusing notation – combining EROs:

$\xrightarrow[R_2\times(2)]{R_1+3R_2}$ We first add $3\times$(Row 2) to Row 1, and then scale (Row 2) by 2.

$A\bar{x}=\bar{b}$ has a solution for every $\bar{b}\in\mathbb{R}^m$ if there is a pivot in every row of $A\in\mathbb{R}^{m\times n}$, i.e., it has m pivots.

Set of all $m\times n$ real matrices

We can answer general questions just based on A, without a particular $\bar{b}$ given. For instance, recall Butch's dilemma between partying and tutoring.

$$\begin{aligned} x_1+x_2&=h \\ 8x_1+16x_2&=d \end{aligned} \qquad \left\{\begin{aligned} h&=\text{\# hrs available} \\ d&=\text{\# dollars available} \end{aligned}\right\}$$

Qn. Can Butch balance partying & tutoring for any h & d?

YES! $\begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix} \xrightarrow{R_2-8R_1} \begin{bmatrix} 1 & 1 \\ 0 & 8 \end{bmatrix}$ pivot in every row.

Homogeneous System of Linear Equations (Section 1.5)

$A\bar{x} = \bar{0}$ is a homogeneous system

all right-hand side entries are zero.

$\bar{x} = \bar{0}$ is always a solution, and hence is called the **trivial solution.** We are interested in finding more interesting solutions.

A non-zero (i.e., at least one entry $\neq 0$) $\bar{x}$ which is a solution is called a **non-trivial solution**.

Prob 2. pg 55

$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

Does this system have a non-trivial solution?

The rhs column always remains at zero, and hence we can ignore it from the augmented matrix.

$$\begin{bmatrix} 1 & -3 & 7 \\ -2 & 1 & -4 \\ 1 & 2 & 9 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & -3 & 7 \\ 0 & -5 & 10 \\ 0 & 5 & 2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -3 & 7 \\ 0 & -5 & 10 \\ 0 & 0 & 12 \end{bmatrix}$$

pivot in every row, and no free variables.

We can use more EROs to go to the reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence the system has a unique solution $x_1 = 0, x_2 = 0, x_3 = 0$, which is the trivial solution. So it does not have any non-trivial solutions.

<u>Note</u>: We need **free variable(s)** to get **non-trivial solutions** to $A\bar{x} = \bar{0}$.

<u>Prob 3, pg 55</u>

Describe the solution set.

$$-3x_1 + 5x_2 - 7x_3 = 0$$
$$-6x_1 + 7x_2 + x_3 = 0$$

$A\bar{x} = \bar{0}$ with $A = \begin{bmatrix} -3 & 5 & -7 \\ -6 & 7 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -3 & 5 & -7 \\ -6 & 7 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} -3 & 5 & -7 \\ 0 & -3 & 15 \end{bmatrix} \xrightarrow[R_2 \times -\frac{1}{3}]{R_1 \times -\frac{1}{3}} \begin{bmatrix} 1 & -5/3 & 7/3 \\ 0 & 1 & -5 \end{bmatrix} \xrightarrow{R_1 + \frac{5}{3}R_2}$$

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \end{bmatrix}$$

x_3 is free

$$x_1 \quad 6x_3 = 0$$
$$x_2 - 5x_3 = 0$$

$$\left.\begin{aligned} x_1 &= 6x_3 \\ x_2 &= 5x_3 \\ x_3 &\text{ free} \end{aligned}\right\} \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} x_3$$

x_3 free.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} x_3$$

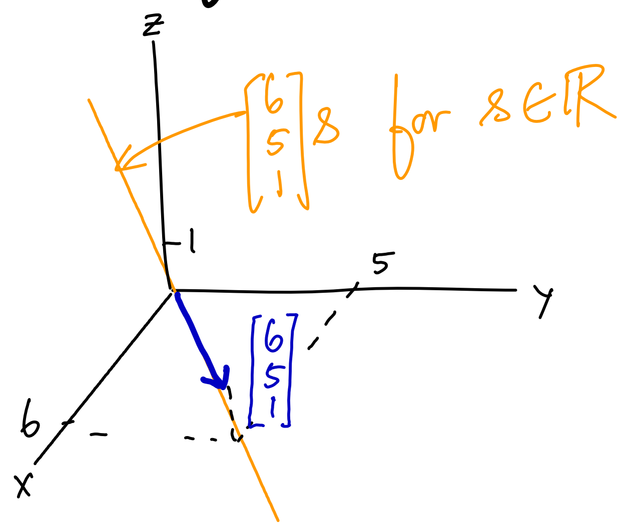
We can write these solutions as

$$\bar{x} = \underbrace{\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} s}_{\text{(vector)} \times \text{(parameter)}}$$

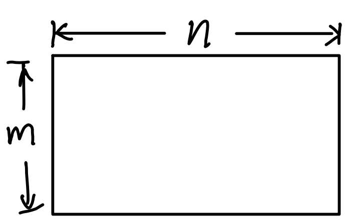
for a parameter s, whose value can be freely set.

This is the **parametric vector form** of all solutions to $A\bar{x} = \bar{0}$

Setting $s=0$ gives the trivial solution.



<u>Detour</u>: Will we always have non-trivial solutions for $A\bar{x} = \bar{0}$, when $n > m$?



$m < n$

The max # pivots A can have is m here. Hence $A\bar{x} = \bar{0}$ has at least one free variable. So there are non-trivial solutions.

Back to the problem ...

Let us consider a non-homogeneous system corresponding to the homogeneous system we just solved.

$$-3x_1+5x_2-7x_3 = 3$$
$$-6x_1+7x_2+x_3 = 7$$

Note: In $A\bar{x}=\bar{b}$, if at least one entry in $\bar{b}$ is $\neq 0$, the system is non-homogeneous.

$$\left[\begin{array}{ccc|c} -3 & 5 & -7 & 3 \\ -6 & 7 & 1 & 7 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} -3 & 5 & -7 & 3 \\ 0 & -3 & 15 & 1 \end{array}\right] \xrightarrow[R_2\times -1/3]{R_1\times -1/3} \left[\begin{array}{ccc|c} 1 & -5/3 & 7/3 & -1 \\ 0 & 1 & -5 & -1/3 \end{array}\right]$$

$$\xrightarrow{R_1+5/3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -6 & -14/9 \\ 0 & 1 & -5 & -1/3 \end{array}\right]$$

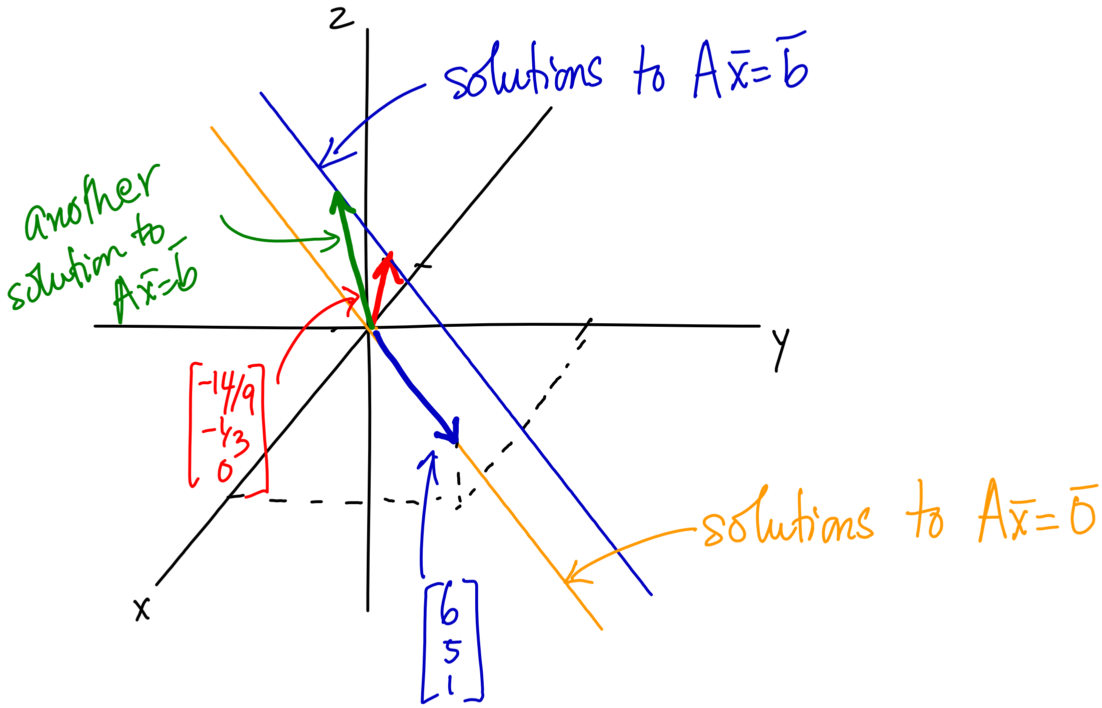
$x_1 = 6x_3 - 14/9$
$x_2 = 5x_3 - 1/3$, x_3 free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{matrix} 6x_3 - 14/9 \\ 5x_3 - 1/3 \\ x_3 \end{matrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -14/9 \\ -1/3 \\ 0 \end{bmatrix} \text{ or}$$

$$\bar{x} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} s + \begin{bmatrix} -14/9 \\ -1/3 \\ 0 \end{bmatrix}, \quad s \in \mathbb{R}$$

is the parametric vector form of the solution.

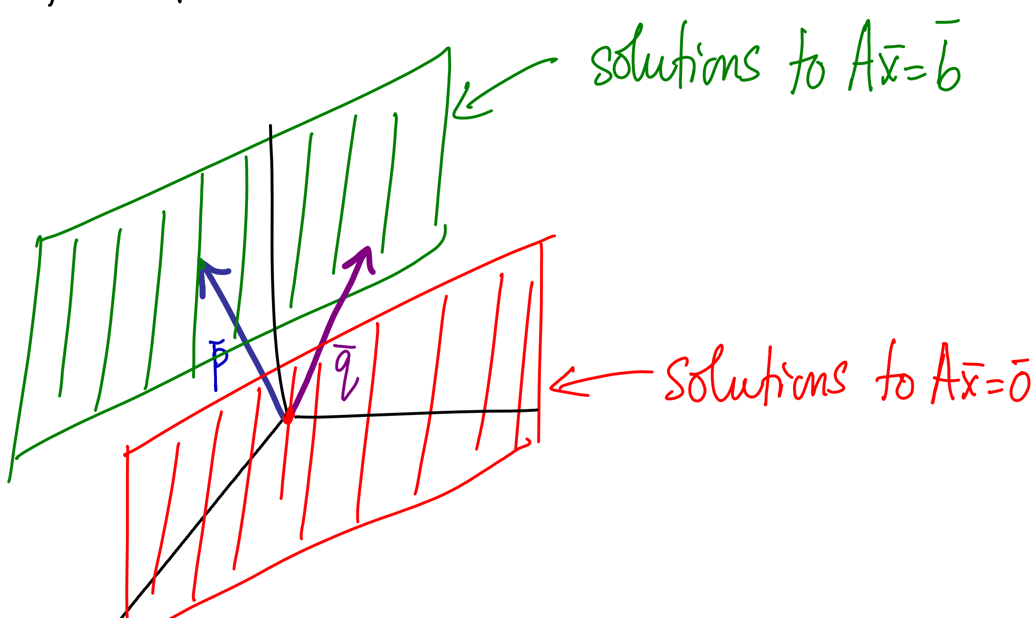
Note that the solutions to the non-homogeneous system is obtained by adding the vector $\begin{bmatrix} -14/9 \\ -1/3 \\ 0 \end{bmatrix}$ to the solutions of the homogeneous system.



Result (Theorem 6 in pg 53). Let $\bar{p}$ be one particular solution to $A\bar{x}=\bar{b}$, i.e., $A\bar{p}=\bar{b}$. Then all solutions to $A\bar{x}=\bar{b}$ are given by $\bar{x}=\bar{p}+\bar{v}_h$, where $\bar{v}_h$ is a solution to $A\bar{x}=\bar{0}$.

<u>Illustration</u>

$\bar{q}$ is another particular solution to $A\bar{x}=\bar{b}$.



MATH 230 - Lecture 7 (02/01/2011)

Homogeneous and non-homogeneous systems

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_1 + 5x_2 + 4x_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \end{bmatrix} \left\} \begin{aligned} x_1 &= -7x_3 \\ x_2 &= 2x_3 \end{aligned} \right.$$

x_3 free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} s, \quad s \in \mathbb{R}$$

$$x_1 + 2x_2 + 3x_3 = 2$$
$$2x_1 + 5x_2 + 4x_3 = -3$$

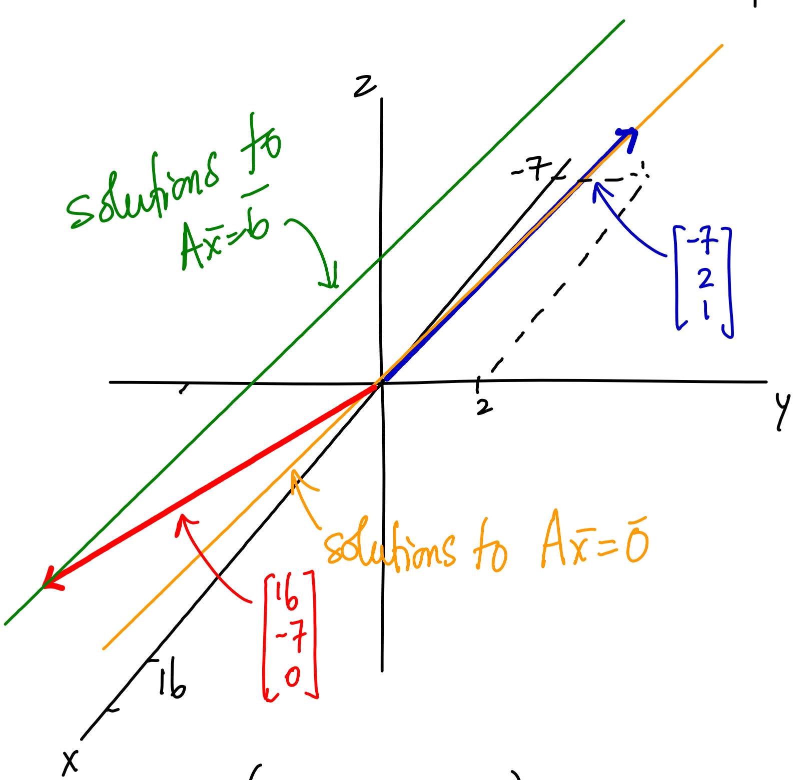
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 5 & 4 & -3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -7 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 16 \\ 0 & 1 & -2 & -7 \end{array}\right]$$

$$x_1 = 16 - 7x_3$$
$$x_2 = -7 + 2x_3$$
x_3 free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ -7 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} s, \quad s \in \mathbb{R}$$

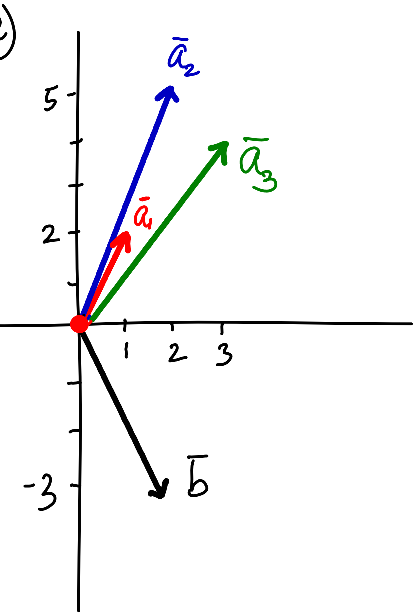
$$\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\bar{a}_1} x_1 + \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{\bar{a}_2} x_2 + \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\bar{a}_3} x_3 = \underbrace{\begin{bmatrix} 2 \\ -3 \end{bmatrix}}_{\bar{b}}$$



(Row picture)

(column picture)

To get to $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$, we first get to origin by taking a linear combination of $\bar{a}_1, \bar{a}_2, \bar{a}_3$, and then jump to $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.



To get to $\bar{b}$ by taking linear combinations of $\bar{a}_1, \bar{a}_2, \bar{a}_3$, we can first get to $\begin{bmatrix}0\\0\end{bmatrix}$ (origin) by taking combination(s) of $\bar{a}_1, \bar{a}_2, \bar{a}_3$, and then just add $\bar{b}$.

Section 1.6 – Applications

Prob 4, pg 63

4. Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M and retains the rest. Sector E sells 30% of its output to A, 35% to M, and 25% to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E, and 30% to M and retains the rest.

a. Construct the exchange table for this economy.

b. [M] Find a set of equilibrium prices for the economy.

The exchange table gives how production, or revenue, is exchanged between the various sectors in the economy. From the exchange table, we should be able to write down linear equations involving the "prices" or "production" from each sector.

Exchange table for economy

output	A	E	M	T	purchased by (or used by)
	0.65	0.30	0.3	0.2	A
	0.10	0.10	0.15	0.1	E
	0.25	0.35	0.15	0.3	M
	0	0.25	0.40	0.4	T

add up to 1. Gives the fractions of the total output from A being used by each sector.

Note that numbers in each column add up to 1.

Let p_A, p_E, p_M, p_T be the prices. For equilibrium, we must have "input" price = "output" price. From the exchange table, we get the following equations.

$$\overset{-0.35}{\cancel{0.65}}\, p_A + 0.3p_E + 0.3p_M + 0.2p_T = \cancel{p_A}\ 0$$

$$0.1p_A + \overset{-0.9}{\cancel{0.1}}\, p_E + 0.15p_M + 0.1p_T = \cancel{p_E}\ 0$$

$$0.25p_A + 0.35p_E + \overset{-0.85}{\cancel{0.15}}\, p_M + 0.3p_T = \cancel{p_M}\ 0$$

$$0p_A + 0.25p_E + 0.4p_M + \overset{-0.6}{\cancel{0.4}}\, p_M = \cancel{p_T}\ 0$$

This is the homogeneous system $A\bar{p} = \bar{0}$, where

$$A = \begin{bmatrix} -0.35 & 0.3 & 0.3 & 0.2 \\ 0.1 & -0.9 & 0.15 & 0.1 \\ 0.25 & 0.35 & -0.85 & 0.3 \\ 0 & 0.25 & 0.4 & -0.6 \end{bmatrix} \quad \text{and} \quad \bar{p} = \begin{bmatrix} p_A \\ p_E \\ p_M \\ p_T \end{bmatrix}.$$

Switched to MATLAB - see course web page for commands and output!

From MATLAB, we get the reduced echelon form of A as

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & -2.028 \\ 0 & 1 & 0 & -0.531 \\ 0 & 0 & 1 & -1.168 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

MATH230 - Lecture 8 (02/03/2011)

Use words carefully!

~~system~~ is unique $\longrightarrow$ solution to the system is unique.

Market equilibrium problem (Prob 4. pg 63)

p_A, p_E, p_M, p_T prices of sectors A, E, M, T. $A\bar{p} = \bar{0}$

Reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & -2.03 \\ 0 & 1 & 0 & -0.53 \\ 0 & 0 & 1 & -1.17 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

p_T is a free variable

2 decimal places will do, as p's are prices

$p_A = 2.03\, p_T$,
$p_E = 0.53\, p_T$,
$p_M = 1.17\, p_T$.

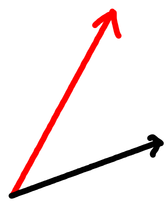
Hence, if $p_T = \$100$, then $p_A = \$203$, $p_E = \$53$, and $p_M = \$117$, are the equilibrium prices.

We could have chosen p_A, or p_E, or p_M as the free variable here! Equivalently, the equilibrium prices can be expressed in terms of any one of the four variables. For instance,

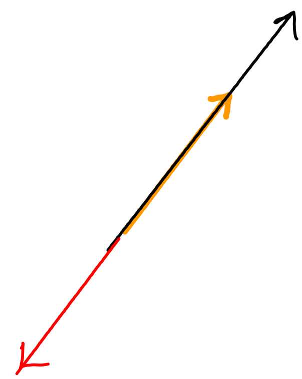
$$p_T = \left(\frac{1}{2.03}\right) p_A. \text{ Hence } p_E = \left(\frac{0.53}{2.03}\right) p_A \text{ and } p_M = \left(\frac{1.17}{2.03}\right) p_A.$$

(See course web page for MATLAB session!)

Linear Independence (Section 1.7)



linearly independent vectors



linearly dependent vectors.

$\{\bar{v}_1, \dots, \bar{v}_n\}$ are vecters in $\mathbb{R}^m$ (i.e., $\bar{v}_j \in \mathbb{R}^m$ for each j).

Def: The (set of) vecters $\{\bar{v}_1, \dots, \bar{v}_n\}$ (is) are linearly independent (LI) if $x_1\bar{v}_1 + \dots + x_n\bar{v}_n = \bar{0}$ has only the trivial solution. Else the vector are linearly dependent (LD).

Equivalently, the columns of a matrix A are LI if and only if $A\bar{x} = \bar{0}$ has only the trivial solution.

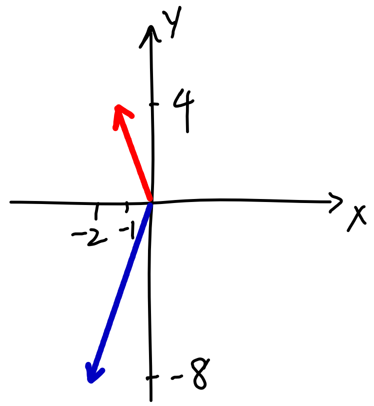
(This happens when there is a pivot in every column of A, i.e., there are no free variables).

Prob 4, Pg 71

$\begin{bmatrix}-1\\4\end{bmatrix}, \begin{bmatrix}-2\\-8\end{bmatrix}$ are the vectors LI? Justify.

$A=\begin{bmatrix}-1 & -2\\4 & -8\end{bmatrix} \xrightarrow{R_2+4R_1} \begin{bmatrix}-1 & -2\\0 & -16\end{bmatrix}$

No free variables. Hence $A\bar{x}=\bar{0}$ has only the trivial solution. So the vectors are LI.



$\longrightarrow$ The vectors are indeed LI!

Prob 12, Pg 71

$\begin{bmatrix}2\\-4\\1\end{bmatrix}, \begin{bmatrix}-6\\7\\-3\end{bmatrix}, \begin{bmatrix}8\\h\\4\end{bmatrix}$ For what h are these vectors LD?

$$A=\begin{bmatrix}2 & -6 & 8\\-4 & 7 & h\\1 & -3 & 4\end{bmatrix} \xrightarrow[R_2+4R_3]{R_1-2R_3} \begin{bmatrix}0 & 0 & 0\\0 & -5 & h+16\\1 & -3 & 4\end{bmatrix} \xrightarrow{R_1 \rightleftharpoons R_3} \begin{bmatrix}1 & -3 & 4\\0 & -5 & h+16\\0 & 0 & 0\end{bmatrix}$$

Every column does not have a pivot (or, x_3 is a free variable). Hence $A\bar{x}=\bar{0}$ has non-trivial solutions. So, the vectors are LD for $h \in \mathbb{R}$.

Special cases of LI/LD of $\{\bar{v}_1, \dots, \bar{v}_n\}$ in $\mathbb{R}^m$

① One vector : $\bar{v}_1 \in \mathbb{R}^m$ is LI, unless $\bar{v}_1 = \bar{0}$

$x_1 \bar{v}_1 = \bar{0}$ only when $x_1 = 0$. But if $\bar{v}_1 = \bar{0}$, then $x \in \mathbb{R}$. Hence $\{\bar{0}\}$ is LD.

② Two vectors $\bar{v}_1, \bar{v}_2$ are LI if they do not lie on the same line. Equivalently, $\bar{v}_1, \bar{v}_2$ are LD if $\bar{v}_1 = c\,\bar{v}_2$ for a scalar c.

MATH 230 - Lecture 9 (02/08/2011)

Special cases of LI/LD vectors

(2) 2 vectors (seen in last lecture).

e.g., $\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$ are LD, as $\bar{v}_2 = 3\bar{v}_1$.

(3) If the set of vectors $\{\bar{v}_1, \dots, \bar{v}_n\}$ contains the zero vector, then it is LD. (Theorem 9, pg 69, DL-LAA).

Say, $\bar{v}_1 = \bar{0}$ (zero vector). Then

$$c_1\bar{v}_1 + 0\cdot\bar{v}_2 + \dots + 0\cdot\bar{v}_n = \bar{0}$$ for $c_1 \neq 0$. As such,

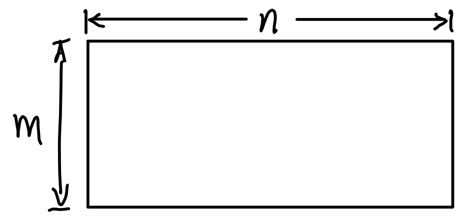
$\bar{x} = \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a non-trivial solution to $A\bar{x} = \bar{0}$ with

$A = [\bar{v}_1 \ \bar{v}_2 \cdots \bar{v}_n]$.

(4) $\{\bar{v}_1, \dots, \bar{v}_n\}$ with $\bar{v}_j \in \mathbb{R}^m$ and $n > m$ is LD.

If there are more vectors than the # entries in each of them, the set is LD. (Theorem 8, pg 69, DL-LAA)

For $A = [\bar{v}_1\ \bar{v}_2 \cdots \bar{v}_n]$, for $\bar{v}_j \in \mathbb{R}^m$ with $m < n$, the matrix looks like



. It must have at least one free variable, as the max. # pivots it can have is m.

<u>Prob 17, pg 71</u> Is the set of vectors LD? Justify.

$$\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$$
$\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3$

This set of vectors is LD, as it contains the zero vector.

Note that $\{\bar{v}_1\}$ as well as $\{\bar{v}_1, \bar{v}_3\}$ here are LI.

$\{\bar{v}_1, \bar{v}_3\}$: $\begin{bmatrix} 3 & -6 \\ 5 & 5 \\ -1 & 4 \end{bmatrix} \xrightarrow[R_2+5R_3]{R_1+3R_3} \begin{bmatrix} 0 & 6 \\ 0 & 25 \\ -1 & 4 \end{bmatrix}$ No free variables. Hence $\{\bar{v}_1, \bar{v}_3\}$ is LI.

But $\{\bar{v}_1, \bar{v}_2\}$ and $\{\bar{v}_2, \bar{v}_3\}$ are both LD, as $\bar{v}_2 = \bar{0}$.

But if a set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LI, then all its subsets (or sub collections of vectors) must be LI.

Theorem 7 (DL-LAA page 68)

$S=\{\bar{v}_1,\ldots,\bar{v}_n\}$ of 2 or more vectors $(n\geq 2)$ in $\mathbb{R}^m$ is LD if and only if one vector in S is a linear combination of the other vectors. In fact, if S is LD and $\bar{v}_1\neq\bar{0}$, then some $\bar{v}_j,\ j>1$ is a linear combination of the preceding vectors $\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_{j-1}$.

e.g., $\bar{v}_3=c_1\bar{v}_1+c_2\bar{v}_2$ or
$\bar{v}_5=c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3+c_4\bar{v}_4$, etc!

Part of proof (Part 2)

$S=\{\bar{v}_1,\ldots,\bar{v}_n\}$ is LD

iff ← (if and only if) or $\Longleftrightarrow$

$$c_1\bar{v}_1+\cdots+c_n\bar{v}_n=\bar{0} \text{ for } \bar{c}=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\neq\bar{0}$$ (at least one $c_j\neq 0$)

Given that $\bar{v}_1\neq 0$, we cannot have $\bar{c}=\begin{bmatrix}c_1\\0\\0\\ \vdots\\0\end{bmatrix}$ as a possible linear combination that gives the zero vector.

In general, we can have $c_1\neq 0, c_2\neq 0,\ldots, c_j\neq 0$, and $c_{j+1}=0, c_{j+2}=0,\ldots,c_n=0$ for a combination that works, for some $j>1$.

We get $c_1\bar{v}_1 + \dots + c_j\bar{v}_j + 0.\bar{v}_{j+1} + \dots + 0.\bar{v}_n = \bar{0}$

$\neq 0 \; \dots \neq 0 \quad \neq 0$

$\Rightarrow \quad c_j\bar{v}_j = -c_1\bar{v}_1 - c_2\bar{v}_2 - \dots - c_{j-1}\bar{v}_{j-1}$

("implies")

As $c_j \neq 0$, dividing by c_j gives

$$\bar{v}_j = \left(\frac{-c_1}{c_j}\right)\bar{v}_1 + \left(\frac{-c_2}{c_j}\right)\bar{v}_2 \dots + \left(\frac{-c_{j-1}}{c_j}\right)\bar{v}_{j-1}$$

Hence $\bar{v}_j$ can be written as a combination of the vectors preceeding it.

<u>Prob 6, Pg 71</u>

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}.$$ Are the columns of A LI?

Check if $A\bar{x} = \bar{0}$ has non-trivial solutions or not.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \xrightarrow[R_4-5R_3]{R_1+4R_3} \begin{bmatrix} 0 & -3 & 12 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 0 & 4 & -9 \end{bmatrix} \xrightarrow[R_4+4R_2]{R_1-3R_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix} \xrightarrow[R_4\to R_3]{R_3\to R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column has a pivot, so no free variables. Hence the columns of A are LI.

Prob 36 Pg 72 True/False "with more justification"

If $\bar{v}_1, \dots, \bar{v}_4$ are in $\mathbb{R}^4$, and $\bar{v}_3$ is not a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_4$, then $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is LI.

The statement is false, as we could have $\bar{v}_4$ being a linear combination of $\bar{v}_1$ and $\bar{v}_2$. Or, $\bar{v}_1$ for instance, could be the zero vector.

If $\underbrace{\bar{v}_4 = c_1\bar{v}_1 + c_2\bar{v}_2}$, then $\bar{x} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ -1 \end{bmatrix}$ is a non-trivial solution to the system $A\bar{x} = \bar{0}$, where $A = [\bar{v}_1\ \bar{v}_2\ \bar{v}_3\ \bar{v}_4]$.

$$c_1\bar{v}_1 + c_2\bar{v}_2 + 0\bar{v}_3 - \bar{v}_4 = \bar{0}$$

Linear Transformation (Section 1.8)

"mappings"

$$A\bar{x} = \bar{b}$$

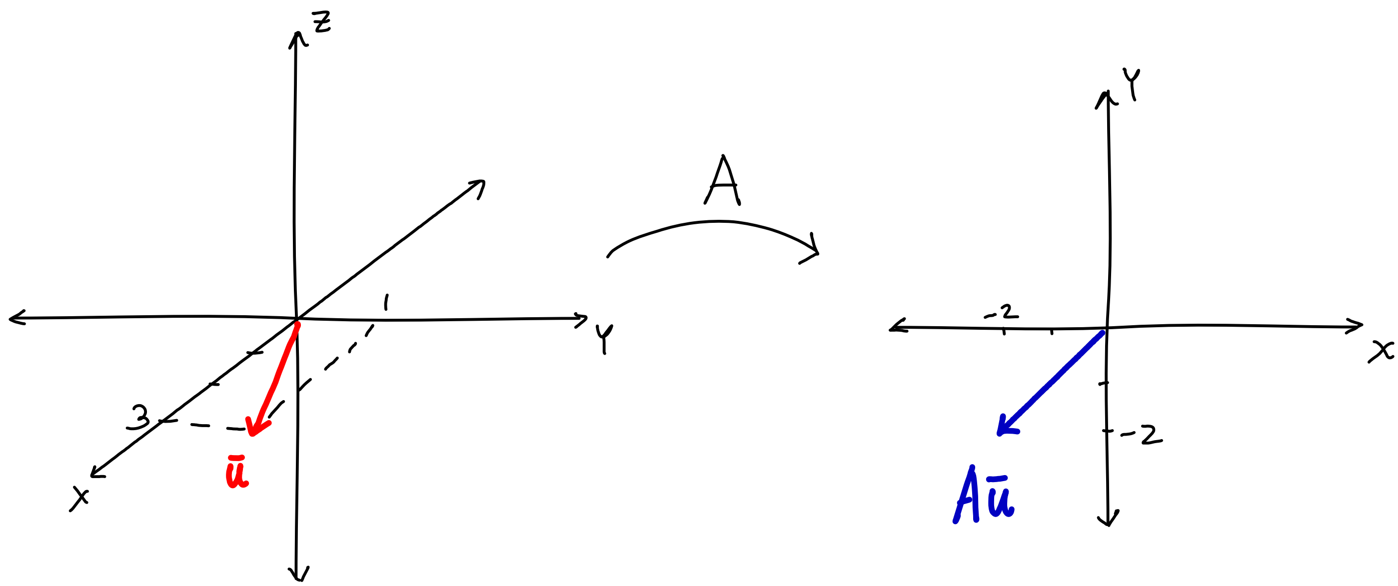
"Hit" $\bar{x}$ with A to get $\bar{b}$

A "maps" $\bar{x}$ to $\bar{b}$

When $\bar{x}$ is "input" to A, $\bar{b}$ is returned.

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$A\bar{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \times 3 + \begin{bmatrix} -5 \\ 7 \end{bmatrix} \times 1 + \begin{bmatrix} -7 \\ 5 \end{bmatrix} \times 0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$



MATH 230 – Lecture 10 (02/10/2011)

A note on writing general solutions in parametric vector form.

e.g., $\bar{x} \in \mathbb{R}^4$ with x_3, x_4 free:

$$\bar{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} x_4, \quad x_3, x_4 \in \mathbb{R} \qquad \text{OR}$$

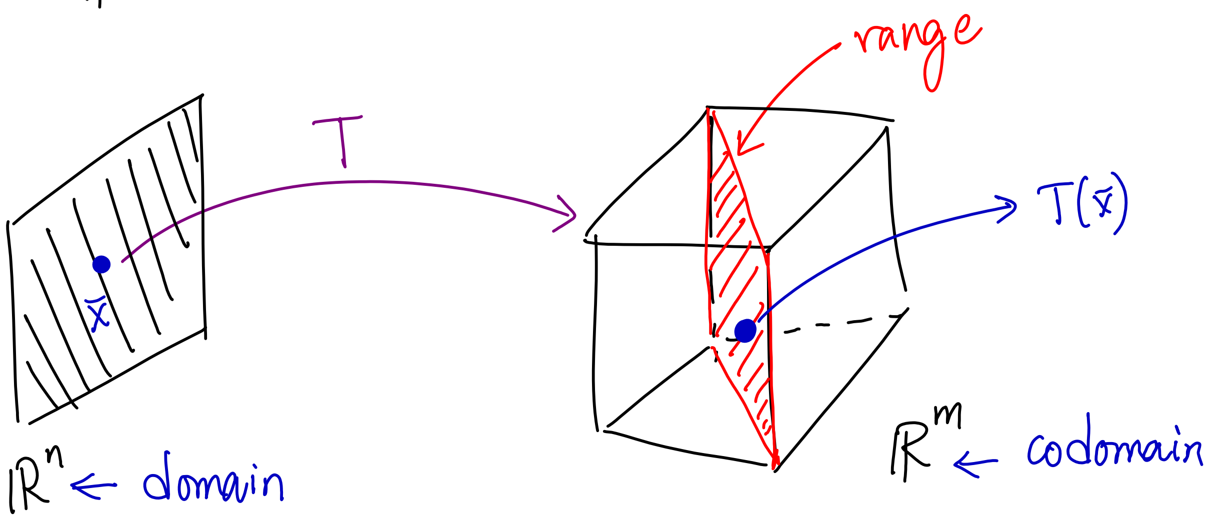
need this specification

$$\bar{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} t, \quad s, t \in \mathbb{R}$$

Back to Linear transformations

Def: A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns every vector $\bar{x}$ in $\mathbb{R}^n$ a vector $T(\bar{x})$ in $\mathbb{R}^m$.

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$



$T(\bar{x})$ is the image of $\bar{x}$ under T.

$$T: \underset{\text{domain}}{\mathbb{R}^n} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^m}$$

Range is the set of all images (of the form $T(\bar{x})$) under T. It is a subset of the codomain; it could be the entire codomain in some cases.

Linear Transformations

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation (LT) if

(i) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v}$ in $\mathbb{R}^n$; and

(ii) $T(c\bar{u}) = cT(\bar{u})$ for all $\bar{u} \in \mathbb{R}^n$, and scalar c.

In words, T is linear if it preserves vector addition and scalar-vector multiplication.

Equivalently, we can say that T is linear if the image of a sum is the sum of images, and if image of scalar $\times$ vector is scalar $\times$ image of vector, under T.

As a consequence of Rules (i) and (ii), we get

$$T(\bar{0}) = \bar{0} \text{ and}$$

$$T(c\bar{u}+d\bar{v}) = cT(\bar{u})+dT(\bar{v}) \text{ for } c,d \in \mathbb{R} \text{ and } \bar{u},\bar{v} \in \mathbb{R}^n.$$

<u>Prob 17, pg 80</u> $T:\mathbb{R}^2 \to \mathbb{R}^2$ is an LT.

$\bar{u} = \begin{bmatrix}5\\2\end{bmatrix}$, $\bar{v} = \begin{bmatrix}1\\3\end{bmatrix}$, $T(\bar{u}) = \begin{bmatrix}2\\1\end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix}-1\\3\end{bmatrix}$. Find the images under T of $3\bar{u}$, $2\bar{v}$, and $3\bar{u}+2\bar{v}$.

$$T(3\bar{u}) = 3T(\bar{u}) = 3\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix}$$

$$T(2\bar{v}) = 2T(\bar{v}) = 2\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$$

$$T(3\bar{u}+2\bar{v}) = 3T(\bar{u})+2T(\bar{v}) = 3\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}3\times2+2\times-1\\3\times1+2\times3\end{bmatrix} = \begin{bmatrix}4\\9\end{bmatrix}.$$

Equivalently, since we just found $T(3\bar{u})$ and $T(2\bar{v})$, we can write

$$T(3\bar{u}+2\bar{v}) = T(3\bar{u})+T(2\bar{v}) = \begin{bmatrix}6\\3\end{bmatrix} + \begin{bmatrix}-2\\6\end{bmatrix} = \begin{bmatrix}4\\9\end{bmatrix}.$$

Prob 24, pg 81

Suppose $\bar{v}_1, \dots, \bar{v}_p$ span $\mathbb{R}^n$, and $T: \mathbb{R}^n \to \mathbb{R}^n$ be an LT. It is known that $T(\bar{v}_i) = \bar{0}$ for all $i = 1, \dots, p$. Show that T is the zero-transformation.

$T(\bar{x}) = \bar{0}$ for all $\bar{x} \in \mathbb{R}^n$.

We want to show that $T(\bar{x}) = \bar{0}$ for all $\bar{x} \in \mathbb{R}^n$.

$\bar{v}_1, \dots, \bar{v}_p$ span $\mathbb{R}^n$. Hence any $\bar{x} \in \mathbb{R}^n$ can be written as $\bar{x} = c_1 \bar{v}_1 + \cdots + c_p \bar{v}_p$ for scalars $c_1, \dots, c_p$.

Hence $T(\bar{x}) = T(c_1 \bar{v}_1 + \cdots + c_p \bar{v}_p)$

could skip.

$$\left.\begin{aligned} &= \cancel{T(c_1 \bar{v}_1) + \cdots + T(c_p \bar{v}_p)} \\ &= c_1 T(\bar{v}_1) + \cdots + c_p T(\bar{v}_p) \end{aligned}\right\} \text{ as } T \text{ is linear}$$

$= c_1 \bar{0} + \cdots + c_p \cdot \bar{0}$ as $T(\bar{v}_i) = \bar{0}$ for all i.

$= \bar{0}$

Prob 32, pg 81

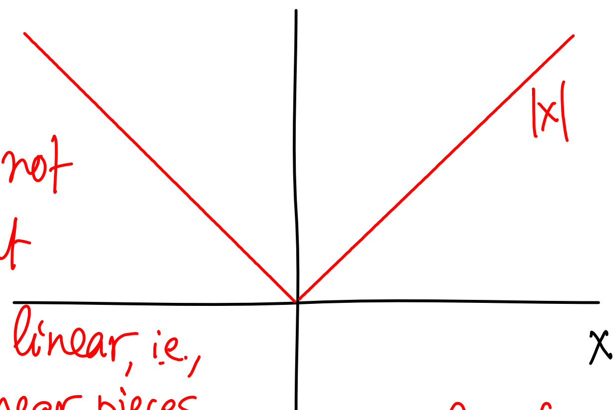
$$T(x_1, x_2) = (4x_1 - 3x_2,\ 3|x_2|)$$

absolute value of x_2

Show that T is not a linear transformation.

Show that T violates one of the properties of LTs.

$|x|$ is not linear, but is piecewise linear, i.e., made of linear pieces.



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Consider

$$\bar{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \bar{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$T(\bar{u}) = (4\times 2 - 3\times 1,\ 3|1|) = (9, 3)$$

$$T(\bar{v}) = (4\times 2 - 3\times -1,\ 3|-1|) = (11, 3)$$

$$\bar{u} + \bar{v} = \begin{bmatrix} 2+2 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

$$T(\bar{u}+\bar{v}) = (4\times 4 - 3\times 0,\ 3|0|) = (16, 0)$$

$$T(\bar{u}) + T(\bar{v}) = \begin{bmatrix} 9+11 \\ 3+3 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \end{bmatrix} \neq T(\bar{u}+\bar{v})$$

Since $T(\bar{u}+\bar{v}) \neq T(\bar{u}) + T(\bar{v})$, T is not an LT.

We could also use a simpler pair of vectors, e.g., $\bar{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, to show $T(\bar{u}) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. $T(\bar{u}) + T(\bar{v}) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, but $T(\bar{u}+\bar{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Matrix of an LT

Result: Every LT T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation $T(\bar{x}) = A\bar{x}$. (Also denoted as $\bar{x} \longmapsto A\bar{x}$).

We can find the A for LT T if we know what T does to the columns of the identity matrix.

$$\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots\ 0 & & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & \ddots & 1 \end{bmatrix}$$

1's along the diagonal and zeros everywhere else.

MATH230 - Lecture 11 (02/15/2011)

Midterm on Tue, Mar 1 (in class).

Topics covered TBA, but should include everything covered up to Lec 12 (Thu, Feb 17).

No MATLAB-related questions!

Practice mid-term will be posted by early next week. Go through homework problems!

Review for midterm on Thursday, Feb 24.

Linear Transformations (LTs)

Prob 12, pg 80

$$A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}.$$

Is $\bar{b}$ in the range of the LT $\bar{x} \longmapsto A\bar{x}$?

Reword: Does $A\bar{x} = \bar{b}$ have at least one solution?

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{array}\right] \xrightarrow[R_4+2R_1]{R_2-R_1} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{array}\right] \xrightarrow[R_4-9R_3]{R_2+3R_3} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & -18 & 11 \end{array}\right]$$

$$\xrightarrow{R_4+6R_2} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 17 \end{array}\right]$$

System is inconsistent. So $\bar{b}$ is not in the range of $\bar{x} \mapsto A\bar{x}$.

Matrix of an LT

"if and only if"

$T: \mathbb{R}^n \to \mathbb{R}^m$ is an LT $\iff$ $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{m \times n}$.

Theorem 10 DL-LAA (pg 83) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be an LT. Then there exists a unique $m \times n$ matrix A such that $T(\bar{x}) = A\bar{x}$ for every $\bar{x} \in \mathbb{R}^n$. This matrix A is given as

$$A = \left[T(\bar{e}_1) \;\; T(\bar{e}_2) \cdots T(\bar{e}_n)\right],$$ where

$\bar{e}_j$ is the j^{th} unit vector in $\mathbb{R}^n$, i.e., $\bar{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← j^{th} position

Proof Let $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. We can write

$$\bar{x} = x_1\bar{e}_1 + x_2\bar{e}_2 + \cdots + x_n\bar{e}_n,$$

where $\bar{e}_j$ is the j^{th} unit vector.

$$\left(= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

So, $T(\bar{x}) = T(x_1\bar{e}_1 + \cdots + x_n\bar{e}_n)$

$= x_1 T(\bar{e}_1) + \cdots + x_n T(\bar{e}_n)$, as T is an LT.

$$= [T(\bar{e}_1) \cdots T(\bar{e}_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$= A\bar{x}$ for $A = [T(\bar{e}_1) \cdots T(\bar{e}_n)]$.

We still need to show that A is unique.

Assume $T(\bar{x}) = B\bar{x}$ for some $m \times n$ matrix B.

So, $T(\bar{e}_j) = B\bar{e}_j = [\bar{b}_1\ \bar{b}_2 \cdots \bar{b}_n] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0\cdot\bar{b}_1 + \cdots + 0\cdot\bar{b}_{j-1} + 1\cdot\bar{b}_j + 0\cdot\bar{b}_{j+1} + \cdots + 0\cdot\bar{b}_n$

j^{th} position

$= \bar{b}_j$

Thus $B = [\bar{b}_1 \cdots \bar{b}_n] = [T(\bar{e}_1) \cdots T(\bar{e}_n)] = A$.

So, A is unique.

e.g., Find the matrix of the LT from $\mathbb{R}^n \to \mathbb{R}^n$ given as $T(\bar{x}) = 3\bar{x}$. → scale the input vector 3 times.

$A = [T(\bar{e}_1)\ T(\bar{e}_2) \cdots T(\bar{e}_n)]$ will be an $n \times n$ matrix.

$$T(\bar{e}_j) = 3\bar{e}_j = 3\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ position}$$

$j = 1, \ldots, n$

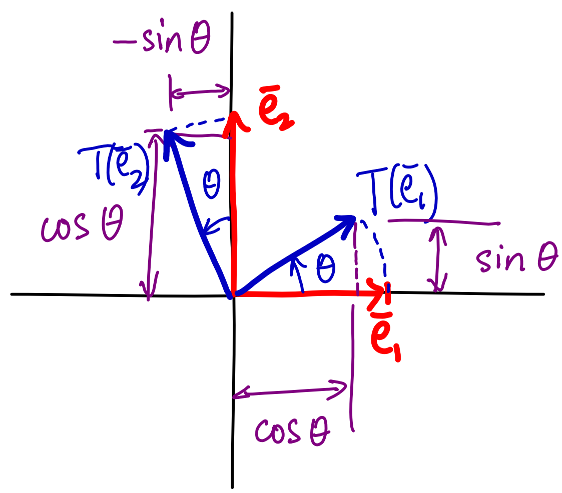
$$A = [3\bar{e}_1\ 3\bar{e}_2 \cdots 3\bar{e}_n] = \begin{bmatrix} 3 & 0 & 0 & \cdots & 0 \\ 0 & 3 & \ddots & & \vdots \\ \vdots & 0 & & \ddots & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & 0 & \cdots & 3 \end{bmatrix} = 3I_n.$$

$n \times n$ identity matrix

Geometric LTs of $\mathbb{R}^2$ (Find A s.t. $T(\bar{x}) = A\bar{x}$ in each case).

1. Rotation (rotate counterclockwise (CCW) by θ degrees).

$$T(\bar{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad T(\bar{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



So, $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

e.g., $\theta = 60°$, $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

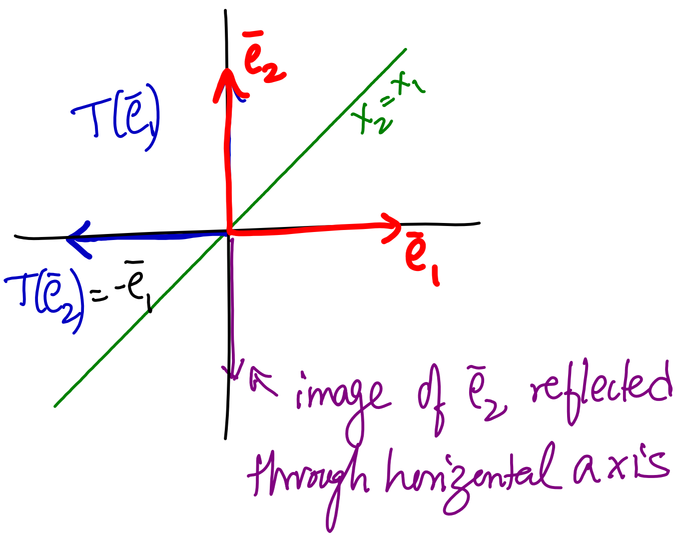
In the "Mangle the Coug" web page, input $A = \begin{bmatrix} 1/2 & -87/100 \\ 87/100 & 1/2 \end{bmatrix}$ (as it only takes fractions).

② Reflections

(Prob 8, Pg 90). Find A of $LT: x \mapsto A\bar{x}$ which reflects on the x-axis and then reflects through the line $x_2 = x_1$.

$T(\bar{e}_1) = \bar{e}_2$ and $T(\bar{e}_2) = -\bar{e}_1$

$$A = [\bar{e}_2 \ -\bar{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

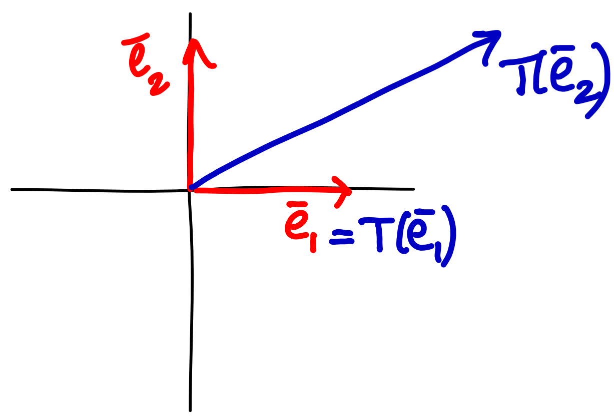


③ Shear

T is a horizontal shear transformation that leaves $\bar{e}_1$ unchanged, and map $\bar{e}_2$ to $\bar{e}_2 + 3\bar{e}_1$.

$$T(\bar{e}_2) = \bar{e}_2 + 3\bar{e}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A = [T(\bar{e}_1) \ T(\bar{e}_2)] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

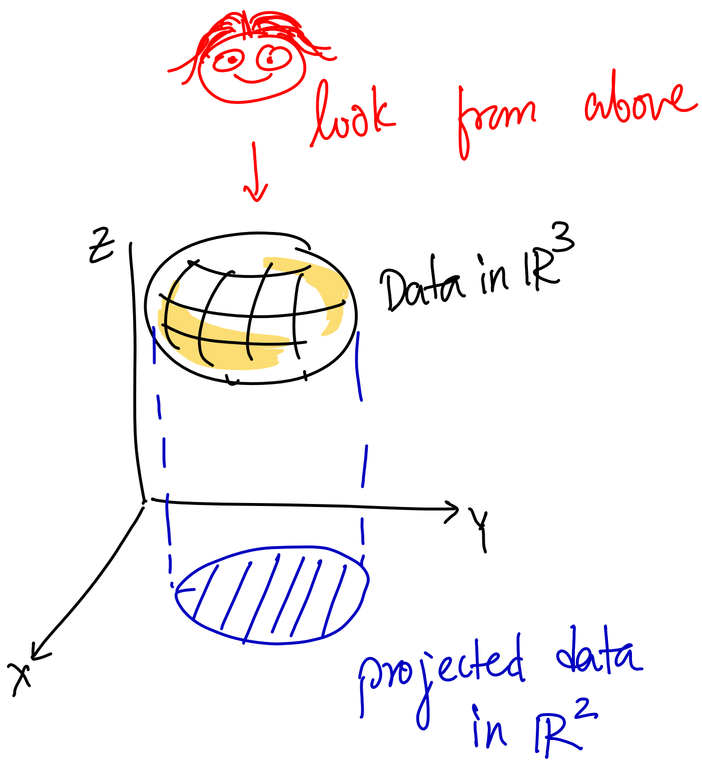


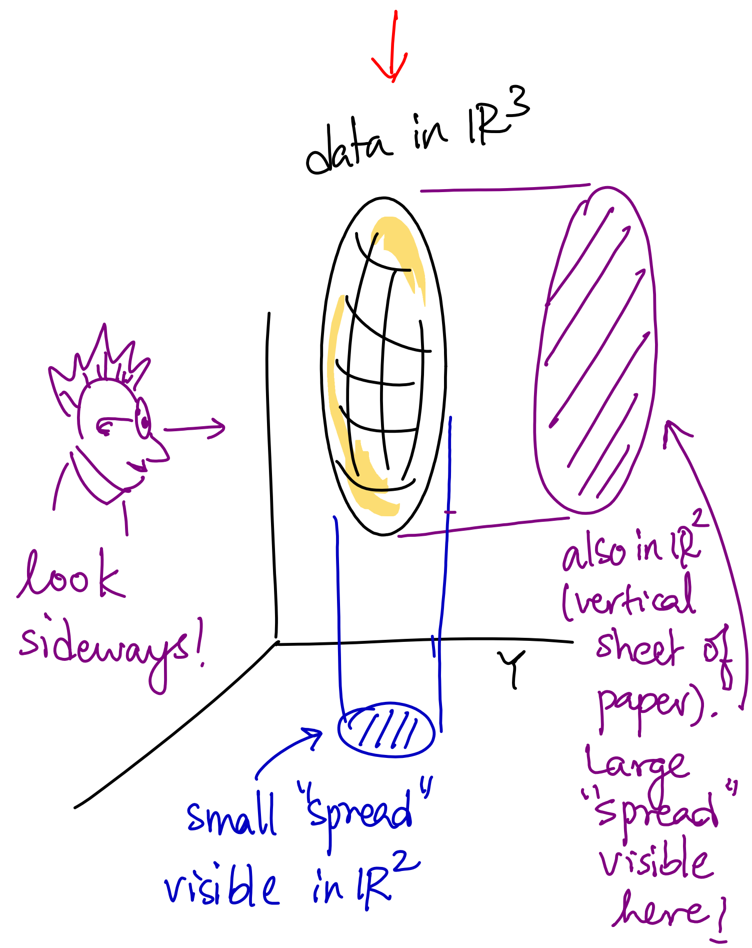
A vertical shear LT will leave $\bar{e}_2$ the same, and transform $\bar{e}_1$ to $\bar{e}_1 + c\bar{e}_2$ for some scalar c.

④ Projection from $\mathbb{R}^n$ to $\mathbb{R}^2$

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. e.g., $\bar{x}$ is part of high-dimensional data.

To visualize better, we could project $\bar{x}$ to $\mathbb{R}^2$.





One such projection $T: \mathbb{R}^n \to \mathbb{R}^2$ is given by

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ignore higher (≥ 3) dimensions

$$A = \left[T(\bar{e}_1) \;\; T(\bar{e}_2) \; \cdots \; T(\bar{e}_n) \right]$$

$$T(\bar{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T(\bar{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T(\bar{e}_j) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for } j \geq 3.$$

$$\bar{e}_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ position}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad 2 \times n \text{ matrix}$$

MATH230 – Lecture 12 (02/17/2011)

Onto and one-to-one transformations (functions or mappings)

Def $T: \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ (domain) **onto** $\mathbb{R}^m$ (codomain) if every $\bar{b} \in \mathbb{R}^m$ is the image under T of **at least** one $\bar{x} \in \mathbb{R}^n$. (i.e., $T(\bar{x}) = \bar{b}$ for at least one $\bar{x}$)

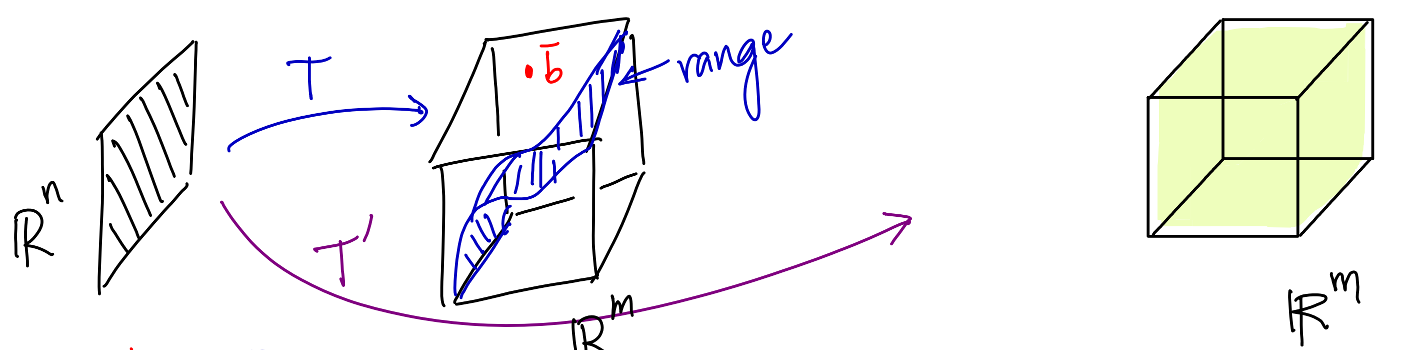
need not be an LT

1 or more

$T: \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each $\bar{b} \in \mathbb{R}^m$ is the image under T of **at most** one $\bar{x} \in \mathbb{R}^n$.

1 or none

($T(\bar{x}) = \bar{b}$ for at most one $\bar{x}$)



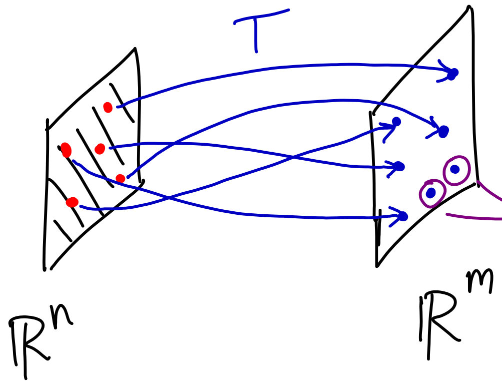
domain

codomain

T is not onto, as the range is strictly smaller than the codomain. (Not every $\bar{b} \in \mathbb{R}^m$ is $T(\bar{x})$ for some $\bar{x} \in \mathbb{R}^n$)

range = codomain

So, T' is onto.



T is one-to-one.

are not images of any $\bar{x} \in \mathbb{R}^n$

LTs that are onto or 1-to-1

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be an LT. So, $T(\bar{x}) = A\bar{x}$ for some $A \in \mathbb{R}^{m \times n}$.

Theorem 12 DL-LAA (pg 89)

1. T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if and only if columns of A **span** $\mathbb{R}^m$. $\big\}$ $A\bar{x} = \bar{b}$ is consistent for all $\bar{b} \in \mathbb{R}^m$.
2. T is **one-to-one** if and only if **columns** of A are linearly independent **(LI)**.

   $A\bar{x} = \bar{b}$ has unique solution or is inconsistent.

Sketch of proof

1. Columns of A span $\mathbb{R}^m \Rightarrow$ every row of A has a pivot, so $A\bar{x} = \bar{b}$ is consistent for every $\bar{b} \in \mathbb{R}^m$. So, there is at least one $\bar{x}$ such that $T(\bar{x}) = A\bar{x} = \bar{b}$.

2. Columns of A are LI $\Rightarrow$ every column has a pivot, or that there are **no free variables.** So $A\bar{x}=\bar{b}$ has at most one solution.

$$\left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{array}\right] \longrightarrow \text{inconsistent system.}$$

$A \quad \bar{b}$

For LTs $\begin{cases} \textbf{pivot in every row} \Rightarrow \textbf{onto} \\ \textbf{pivot in every column} \Rightarrow \textbf{1-to-1} \end{cases}$

Prob 29, pg 91

$T: \mathbb{R}^3 \to \mathbb{R}^4$ is a 1-to-1 LT. Describe all possible echelon forms of the matrix of this LT.

$T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{4\times3}$. T is 1-to-1 means every column of A has a pivot.

$$A = \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$ is the only echelon form possible.

Prob 30, pg 91 $T: \mathbb{R}^4 \to \mathbb{R}^3$ is an onto LT.
Describe all possible echelon forms of A such that $T(\bar{x}) = A\bar{x}$.

A is a 3×4 matrix, and should have a pivot in every row.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

are the possible echelon forms.

In general, if $T: \mathbb{R}^n \to \mathbb{R}^m$ is an LT, $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{m\times n}$.

* T is onto $\Longrightarrow$ $m \le n$ → need pivot in every row; cannot have more rows than columns.
* T is one-to-one $\Longrightarrow$ $m \ge n$ → need pivot in every column; cannot have more columns than rows
* T is one-to-one AND onto $\Longrightarrow$ $m = n$

pivot in every row $\Rightarrow$ consistent for all $\bar{b}$
pivot in every column $\Rightarrow$ no free vars.

$\Bigg\}$ Has a unique solution for each $\bar{b}$.

Prob 23, pg 91 (TRUE/FALSE)

(d) A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if every vector $\bar{x} \in \mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.

FALSE. We need every vector $\bar{b}$ in $\mathbb{R}^m$ to be mapped from ~~some~~ vector $\bar{x} \in \mathbb{R}^n$, not the other way around.

(e) If A is a 3x2 matrix, then the transformation $\bar{x} \longmapsto A\bar{x}$ cannot be 1-to-1.

FALSE. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ gives a 1-to-1 map.

24(e) A is 3x2, then transformation $\bar{x} \mapsto A\bar{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

TRUE. $m > n$ here, so we cannot have a pivot in each row.

Applications (Section 1.10)

Difference Equations

Prob 10, pg 101

10. In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 3% of the suburban population moves into the city. In 2000, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where $\mathbf{x}_0$ is the initial population in 2000. Then estimate the population in the city and in the suburbs two years later, in 2002.

Let c_k = city population in year k

s_k = suburbs population in year k

$k=0$ at 2000, $k=1$ at 2001, and so on.

$\bar{x}_k = \begin{bmatrix} c_k \\ s_k \end{bmatrix}$ vector of populations in year k.

$c_0 = 800{,}000$, $s_0 = 500{,}000$ (starting populations).

Hence $\bar{x}_0 = \begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix}$.

$$c_{k+1} = \underbrace{(1-0.07)}_{\text{fraction of people who stayed put in the city}} c_k + \underbrace{0.03}_{\text{fraction of people from suburbs who moved in}} s_k$$

$$s_{k+1} = (1-0.03)s_k + 0.07c_k$$

similar arguments for suburbs

$$\bar{x}_{k+1} = \begin{bmatrix} c_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{matrix} 0.93c_k + 0.03s_k \\ 0.07c_k + 0.97s_k \end{matrix} = \underbrace{\begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix}}_{M \text{ migration matrix}} \begin{bmatrix} c_k \\ s_k \end{bmatrix}$$

$$\bar{x}_{k+1} = M\bar{x}_k \quad \text{for} \quad M = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix}$$

add to 1   add to 1.

each column adds to 1.

To find populations in 2002, we find $\bar{x}_1$ (populations in 2001), and then $\bar{x}_2$.

(from MATLAB)

$$\bar{x}_1 = M\bar{x}_0 = \begin{bmatrix} 759000 \\ 541000 \end{bmatrix}; \quad \bar{x}_2 = M\bar{x}_1 = \begin{bmatrix} 722100 \\ 577900 \end{bmatrix}.$$

MATH230 – Lecture 13 (02/22/2011)

Matrix Operations (Section 2.1)

$$\underset{m\times n}{A} = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

$\bar{a}_j \in \mathbb{R}^m$

i^{th} row

j^{th} column

A_{ij} is used sometimes in place of a_{ij}.

Notation: uppercase letters stand for matrices,
e.g., $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times n}$ etc.

Matrix Addition

$A+B$ is defined when A and B have the same size. If $C = A+B$, then

$$c_{ij} = a_{ij} + b_{ij} \quad (\text{or } C_{ij} = A_{ij} + B_{ij}).$$

(add corresponding entries)

Scalar multiplication

$B = rA$ for scalar r, then

$$b_{ij} = r \cdot a_{ij}$$ (multiply each entry by scalar r).

Properties of matrix addition & scalar multiplication

(Theorem 1, pg 108).

$A, B, C \in \mathbb{R}^{m\times n}, \quad r, s \in \mathbb{R}$

(a) $A+B = B+A$

(b) $(A+B)+C = A+(B+C)$

(c) $A + 0 = A$ mxn zero matrix

(d) $r(A+B) = rA + rB$

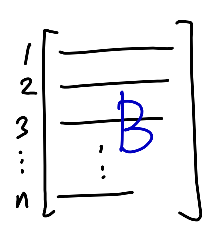
(e) $(r+s)A = rA + sA$

(f) $r(sA) = (rs)A$

Matrix multiplication

We have seen $A\bar{x}$ for $m\times n$ matrix A and n-vector $\bar{x}$, or an $n\times 1$ matrix $\bar{x}$.

Def $C = AB$ is defined when A is $m\times n$ and B is $n\times p$, i.e., when # columns in A = # rows in B.

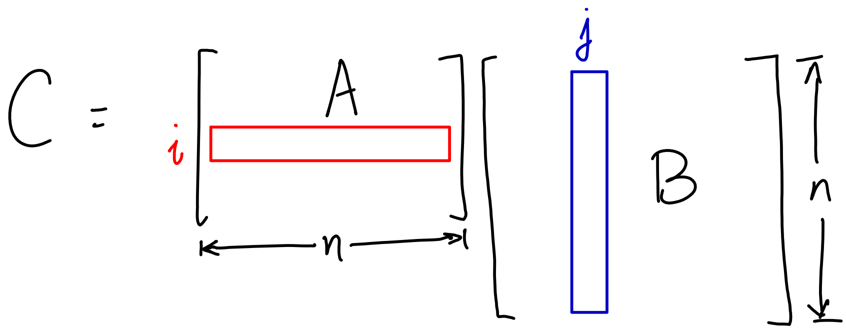
$A = [\bar{a}_1 \cdots \bar{a}_n]$ $\quad B = [\bar{b}_1 \cdots \bar{b}_p] \rightarrow$ 

$\bar{a}_j \in \mathbb{R}^m$ $\quad \bar{b}_i \in \mathbb{R}^n$

need to be same.

$$C = AB = A\left[\bar{b}_1 \cdots \bar{b}_p\right], \quad \bar{b}_i \in \mathbb{R}^n$$

$$= \left[A\bar{b}_1 \;\; A\bar{b}_2 \cdots A\bar{b}_p\right]$$



$$c_{ij} = \boxed{a_{i1}\; a_{i2} \cdots \cdots a_{in}}\;\boxed{\begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{matrix}} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}}$$

Example

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 4 \end{bmatrix}$$

2×2 $\qquad$ 2×3

$C = AB$ is defined, but BA is not defined.

$$C = \left[\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right] = \begin{bmatrix} \boxed{3} & 4 & -4 \\ 3 & \boxed{9} & 0 \end{bmatrix}$$

$2\times 1 + (-1)\times(-1)$

$3\times 3 + 0\times 2$

$C = AB$ for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ will have size $m \times p$, i.e., $C \in \mathbb{R}^{m \times p}$.

Prob 10, pg 116

$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Is $AB = AC$?

What about $AB \stackrel{?}{=} BA$?

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 2\times 8 - 3\times 5 & 2\times 4 - 3\times 5 \\ -4\times 8 + 6\times 5 & -4\times 4 + 6\times 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2\times 5 - 3\times 3 & 2\times -2 - 3\times 1 \\ -4\times 5 + 6\times 3 & -4\times -2 + 6\times 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

So, $AB = AC$.

$$BA = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 8\times 2 + 4\times -4 & 8\times -3 + 4\times 6 \\ 5\times 2 - 5\times 4 & 5\times -3 + 5\times 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -10 & 15 \end{bmatrix}$$

$AB \neq BA$ here.

In general, $AB \neq BA$. In fact, BA may not even be defined. If $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 3}$, then AB is defined, but BA is not.

Properties of matrix multiplication

(Theorem 2, pg 113)

* AB is not typically equal to BA (BA may not even be defined)

(a) $A(BC) = (AB)C$ (associativity)

(b) $A(B+C) = AB+AC$ (left distributive property)

(c) $(B+C)A = BA+CA$ (right distributive property)

(d) $r(AB) = (rA)B = A(rB)$

(e) $I_m A = A = AI_n$ (identity)

I_m : $m\times m$ identity, I_n : $n\times n$ identity matrix

---

Proof of (b). $A(B+C) = AB+AC$

$A_{m\times n}$

$B_{n\times p}$

$C_{n\times p}$

$B = [\bar{b}_1\ \bar{b}_2 \cdots \bar{b}_p]$

$C = [\bar{c}_1\ \bar{c}_2 \cdots \bar{c}_p]$

$$A(B+C) = A\left([\bar{b}_1\,\bar{b}_2\cdots\bar{b}_p] + [\bar{c}_1\,\bar{c}_2\cdots\bar{c}_p]\right)$$

$$= A\left[(\bar{b}_1+\bar{c}_1)\ (\bar{b}_2+\bar{c}_2)\ \cdots\ (\bar{b}_p+\bar{c}_p)\right]$$

from matrix addition of B & C

$$= \left[A(\bar{b}_1+\bar{c}_1)\ A(\bar{b}_2+\bar{c}_2)\cdots A(\bar{b}_p+\bar{c}_p)\right]$$

$$= [A\bar{b}_1 + A\bar{c}_1 \quad A\bar{b}_2 + A\bar{c}_2 \cdots A\bar{b}_p + A\bar{c}_p]$$ from matrix- vector multiplication

$$= [A\bar{b}_1 \; A\bar{b}_2 \cdots A\bar{b}_p] + [A\bar{c}_1 \; A\bar{c}_2 \cdots A\bar{c}_p]$$

$$= AB + AC$$

Transpose of a matrix A (A^T)

T in superscript

Interchange rows and columns of A to get A^T.

e.g. $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -3 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 4 \\ 1 & -3 \\ 0 & 6 \end{bmatrix}$

2×3 $\quad$ 3×2.

If $A \in \mathbb{R}^{m\times n}$, then $A^T \in \mathbb{R}^{n\times m}$.

Properties of matrix transposes

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$

transpose of product = product of transposes in reverse order

Prob 10, pg 116 (contd...)

$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$. Verify that $(AB)^T = B^T A^T$.

$AB = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$ (from before - see page 13-4)

$$\Rightarrow (AB)^T = \begin{bmatrix} 1 & -2 \\ -7 & 14 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 8 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -7 & 14 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 8 & 5 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 0 & 15 \end{bmatrix} = (BA)^T$$

Power of a matrix

If $A \in \mathbb{R}^{n \times n}$, then $A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$

e.g.,

When $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 16 & -24 \\ -32 & 48 \end{bmatrix}$.

We will do an example that motivates the definition of the "inverse" of a matrix.

Prob 17, pg 117

If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$, and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, find B.

Since A is 2×2, and AB 2×3, B must be a 2×3 matrix.

Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$.

$$AB = \overset{A}{\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}} \begin{array}{c} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \\ \begin{array}{ccc} \bar{x} & \phantom{b_{12}} & \phantom{b_{13}} \end{array} \end{array} = \begin{bmatrix} b_{11}-2b_{21} & b_{12}-2b_{22} & b_{13}-2b_{23} \\ -2b_{11}+5b_{21} & -2b_{12}+5b_{22} & -2b_{13}+5b_{23} \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} \\ \begin{array}{ccc} \bar{c} & & \end{array} \end{array}$$

We can find $\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$ by solving the system of linear equations given as $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Similar calculations could give us $\bar{b}_2 = \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix}$ and $\bar{b}_3 = \begin{bmatrix} b_{31} \\ b_{32} \end{bmatrix}$. Notice that the coefficient matrix A is the same in all three systems here!

MATH 230 – Lecture 14 (02/24/2011)

$$AB = \overset{A}{\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

$\bar{b}_1 \quad \bar{b}_2 \qquad \bar{c}_1 \quad \bar{c}_2$

$A\bar{b}_1 = \bar{c}_1$ is a system of two equations in two variables.

$$\underset{A \quad \bar{c}_1}{\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right]} \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]$$

Similarly, we can solve $A\bar{b}_2 = \bar{c}_2$ and $A\bar{b}_3 = \bar{c}_3$. Notice that the EROs are the same as in the case of $A\bar{b}_1 = \bar{c}_1$. We could "club" the three augmented matrices, and do the same EROs in one go.

$$\underset{A \qquad \bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3}{\left[\begin{array}{cc|ccc} 1 & -2 & -1 & 2 & -1 \\ -2 & 5 & 6 & -9 & 3 \end{array}\right]} \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|ccc} 1 & -2 & -1 & 2 & -1 \\ 0 & 1 & 4 & -5 & 1 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{cc|ccc} 1 & 0 & 7 & -8 & 1 \\ 0 & 1 & 4 & -5 & 1 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{B}$

Hence $B = \begin{bmatrix} 7 & -8 & 1 \\ 4 & -5 & 1 \end{bmatrix}$.

With $a, b \in \mathbb{R}$, let $ab = c$. If you are given a and c, to find b, we do, assuming $a \neq 0$,

$$a \cdot b = c$$
$$b = \frac{1}{a} c = a^{-1} \cdot c$$

The previous example had

$AB = C$ (C being the product AB given).

Can we write $B = A^{-1}C$?

We define the inverse of a matrix.

(Section 2.2)

If $A \in \mathbb{R}^{n \times n}$, and if there is a matrix $B \in \mathbb{R}^{n \times n}$ such that $AB = BA = I_n$, then B is called the **inverse** of A, and is denoted as $B = A^{-1}$.

If A^{-1} exists, we say that A is **invertible**.

e.g., $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

Check: $AA^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{-1}A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Theorem 4, DL-LAA pg 119

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A^{-1} exists when $ad-bc \neq 0$.

If $ad-bc \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The quantity $ad-bc$ is called the **determinant** of A. It holds in general (not just for 2x2 matrices) that A is invertible when its determinant is not zero.

$ax=b,\ a\neq 0 \Rightarrow x = \frac{1}{a}b = a^{-1}b$

Theorem 5, DL-LAA page 120

If A is invertible, then $A\bar{x}=\bar{b}$ has a unique solution for any b, given by $\bar{x} = A^{-1}\bar{b}$.

In the previous example ($AB=C$, find B), we could have just calculated B as $A^{-1}C$.

$$A^{-1}C = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -8 & 1 \\ 4 & -5 & 1 \end{bmatrix}.$$

Review for midterm

Hw6. Prob 2

Project from $\mathbb{R}^4$ to $\mathbb{R}^2$ by ignoring 3^{rd} & 4^{th} coordinates, then rotate CCW by 45°.

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \cancel{0} \\ \cancel{0} \end{bmatrix} \xrightarrow[\text{to } \mathbb{R}^2]{\text{project}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow[\text{CCW } 45^\circ]{\text{rotate}} \begin{bmatrix} \cos 45 \\ \sin 45 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \cancel{0} \\ \cancel{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -\sin 45 \\ \cos 45 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\bar{e}_3 = \begin{bmatrix} 0 \\ 0 \\ \cancel{1} \\ \cancel{0} \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ stays same post rotation

$\bar{e}_4 = \begin{bmatrix} 0 \\ 0 \\ \cancel{0} \\ \cancel{1} \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ stays same post rotation

The matrix of LT $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix}$.

Practice Midterm

#6 (Prob 14, DL-LAA page 103)

$\left\{ \begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ a+2 \end{bmatrix} \right\}$ For what a are these vectors LI?

$A = \begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix}$. A must have a pivot in each column for its columns to be LI.

$$\begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix} \xrightarrow{R_2 - aR_1} \begin{bmatrix} 1 & a \\ 0 & a+2-a^2 \end{bmatrix} \neq 0$$

$a+2-a^2=0$ gives $a^2-a-2=0$, i.e., $(a-2)(a+1)=0$

So $a+2-a^2=0$ for $a=2,-1$. Hence the given two vectors are LI for $\underbrace{a \in \mathbb{R} / \{2,-1\}}_{\text{all real values except } 2,-1.}$

or just say $a \neq 2,-1$.

#8 True/False

(a) FALSE.

A does not have a pivot in every row. Since A is square, it does not have a pivot in every column, so there exist(s) free variable(s).

(b) FALSE.

We get the solutions to $A\bar{x}=\bar{b}$ by adding some vector to that of $A\bar{x}=\bar{0}$, not necessarily $\bar{b}$.

(See answers to Prob #1. If A is 3x4, then $\bar{b} \in \mathbb{R}^3$, but the vector you add will be in $\mathbb{R}^4$).

(c) FALSE. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is LD, but has 1 vector with 2 entries.

Only the converse is true, i.e., if there are more vectors than entries in vectors, the set is LD.

(d) FALSE. $\bar{x} \mapsto A\bar{x}$ where $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A has a pivot in every column, but not in every row.

#5. $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T(\bar{e}_1) = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$, $T(\bar{e}_2) = \begin{bmatrix} 3 \\ k \\ 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix} \xrightarrow[R_3 - hR_2]{R_1 - 2R_2} \begin{bmatrix} 0 & 3-2k \\ 1 & k \\ 0 & -hk \end{bmatrix} \xrightarrow{R_1 \rightleftarrows R_2} \begin{bmatrix} 1 & k \\ 0 & 3-2k \\ 0 & -hk \end{bmatrix}$$

Both columns of A have a pivot if either
$3-2k \neq 0$ or $-hk \neq 0$, i.e,
$k \neq \frac{3}{2}$ or $h \neq 0, k \neq 0$. So T is 1-to-1 for
$h \in \mathbb{R}, k \in \mathbb{R}/\{3/2\}$ and $h, k \in \mathbb{R}/\{0\}$.

T cannot be onto, as A cannot have a pivot in every row, i.e, T is onto for <u>NO</u> h & k.

#3. If $A_{3\times4}$ has a pivot in every row, then $\bar{b}$ is in span of columns of A for all $\bar{b} \in \mathbb{R}^3$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{to convert 0's to nonzeros}]{\text{Use EROs}}$$

MATH230 – Lecture 16 (03/03/2011)

Offer to possibly get A in MATH 230:

* If you score 95 or more in the Final, your final score will replace your mid-term score.
* If you score ≥ 90 in the Final, the weights for midterm and final would be changed to 15% and 40%, respectively.

Inverse of an $n\times n$ matrix

If $A \in \mathbb{R}^{n\times n}$, and if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$, then $A^{-1} = B$. (A is invertible, B is the inverse of A; A is the inverse of B).

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is Invertible if $\det A = ad - bc \neq 0$ (determinant). Then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 4, DL-LAA page 119

If $A \in \mathbb{R}^{n\times n}$ is invertible, then $A\bar{x}=\bar{b}$ has a unique solution for any $\bar{b} \in \mathbb{R}^n$ given by $\bar{x} = A^{-1}\bar{b}$.

Theorem 5, DL-LAA page 120.

Prob 6, pg 126

$$8x_1 + 5x_2 = -9$$
$$-7x_1 - 5x_2 = 11$$

Use matrix inverse to solve the system.

$A\bar{x}=\bar{b}$ where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$.

$$\det A = 8\times -5 - (-7)\times 5$$
$$= -40+35 = -5 \neq 0$$

So, A^{-1} exists. $A^{-1} = \frac{1}{-5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}$.

Hence the unique solution is $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\bar{b} = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

In MATLAB, the function **inv()** gives the inverse of a matrix.

e.g., `>> Ainv = inv(A)`

Properties of matrix inverses ($A, B \in \mathbb{R}^{n\times n}$, invertible)

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ → inverse of product = product of inverses in reverse order
3. $(A^T)^{-1} = (A^{-1})^T$

Proof sketch

2. $(AB)^{-1} = B^{-1}A^{-1}$

We want $C \in \mathbb{R}^{n\times n}$ such that

$C(AB) = (AB)C = I$.

Can just verify that $C = B^{-1}A^{-1}$ works.

$$(B^{-1}\underbrace{A^{-1})(A}B) = B^{-1}(A^{-1}A)B = B^{-1}\underbrace{IB} = B^{-1}B = I$$

$$(A\underbrace{B)(B^{-1}}A^{-1}) = A(BB^{-1})A^{-1} = \underbrace{AI}A^{-1} = AA^{-1} = I$$

3. $(A^T)^{-1} = (A^{-1})^T$. We want C such that $CA^T = A^TC = I$.

$C = (A^{-1})^T$ works!
$$\underbrace{(A^{-1})^TA^T = (AA^{-1})^T} = I^T = I$$

$$\underbrace{A^T(A^{-1})^T = (A^{-1}A)^T} = I^T = I.$$

as $(AB)^T = B^TA^T$

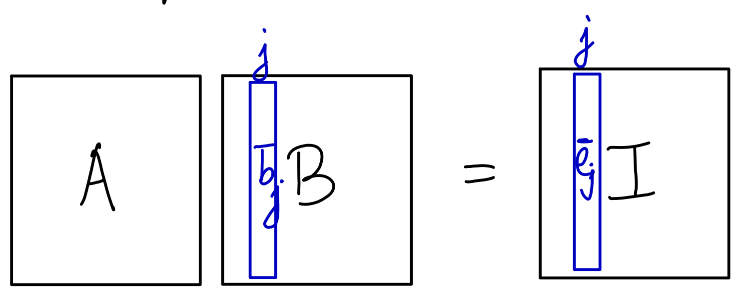
How to invert an $n \times n$ matrix? $(n \geq 3)$

Say we want to find $B = A^{-1}$. We have $AB = I$.

We know how to solve $A\bar{x} = \bar{b}$

$$[A \mid \bar{b}] \xrightarrow{\text{EROs}} \text{rref}([A \mid \bar{b}]).$$

We could just solve for each column of B as a system of n equations in n variables.



For $\bar{b}_j$ (j^{th} column of B), we solve

$$A\bar{b}_j = \bar{e}_j$$ $\rightarrow$ the j^{th} unit vector.

The augmented matrix for this system is $[A \mid \bar{e}_j]$. We could combine the augmented matrices for all j, and look at $[A \mid I]$, and perform EROs on it.

We'll get $[A \mid I] \xrightarrow{\text{EROs}} [I \mid A^{-1}]$ if A^{-1} exists.

Therem 7 DL-LAA pg 123

$A \in \mathbb{R}^{n\times n}$ is invertible if and only if it is row-equivalent to I_n (i.e, you can reduce A to I_n using EROs). In this case, the same EROs reduce I_n to A^{-1}! So, $\text{rref}([A|I_n]) = [I_n|A^{-1}]$

If we do not get I_n in place of A, then A is not invertible.

Prob 32 pg 127

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

Find A^{-1} if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-4R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 2 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$$

A is not invertible, as $A \not\sim I_3$.

→ not row-equivalent

"$\sim$": row-equivalent

Prob 14 pg 126

$B, C \in \mathbb{R}^{m\times n}$, D is invertible, and $(B-C)D = O$.

Show that $B=C$.

Since $(B-C)D$ is defined, D has n row. But D is invertible, so D is $n\times n$.

$$\underbrace{(B-C)}_{m\times n}\underbrace{D}_{n\times n} = \underbrace{O}_{m\times n}$$

D^{-1} exists. Multiply the above equation on the right by D^{-1}!

$$(B-C)DD^{-1} = OD^{-1}$$

$$\Rightarrow (B-C)(DD^{-1}) = OD^{-1} = O$$

$$\Rightarrow (B-C)I = O, \text{ i.e., } B-C = O$$

$$\Rightarrow B=C.$$

Prob 18, pg 126

P is invertible, $A = PBP^{-1}$. Solve for B.

$$(A = PBP^{-1})P$$

$$AP = PB\underbrace{P^{-1}P} = PBI = PB$$

Multiply on left with P^{-1} to get

$$P^{-1}AP = \underbrace{P^{-1}P}B = IB = B.$$

So $\quad B = P^{-1}AP$

For $A, B \in \mathbb{R}^{n\times n}$, if there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that $A = P^{-1}BP$, then A and B are **similar**. Then $B = PAP^{-1}$.

Prob 20, pg 126

$A, B, X \in \mathbb{R}^{n\times n}$, A, X and $(A-AX)$ are invertible, and

$$(A-AX)^{-1} = X^{-1}B \quad \text{———} \quad (1)$$

(a) Is B invertible?

(b) Solve for X from (1).

MATH230 – Lecture 17 (03/08/2011)

Properties of invertible matrices

Prob 20, from Pg 126: A, X, and $(A-AX)$ are invertible. $A, B, X \in \mathbb{R}^{n\times n}$.

$$(A-AX)^{-1} = X^{-1}B \quad \text{——— (1)}$$

(a) Is B invertible?

(b) Solve for X from (1).

$$(A-AX)^{-1} = X^{-1}B$$

$$\Rightarrow X(A-AX)^{-1} = X(X^{-1}B) \qquad X \in \mathbb{R}^{n\times n},\ (A-AX)^{-1} \text{ is also } n\times n$$

$$= IB = B \quad \text{as } A(BC) = (AB)C$$

$$\Rightarrow B^{-1} = [X(A-AX)^{-1}]^{-1} = ((A-AX)^{-1})^{-1}X^{-1} \quad \text{as } (AB)^{-1} = B^{-1}A^{-1}$$

$$= (A-AX)X^{-1} \quad \text{as } (A^{-1})^{-1} = A$$

So B^{-1} exists, showing that B is invertible.

(b) $(A-AX)^{-1} = X^{-1}B \rightsquigarrow \dfrac{1}{a-ax} = \dfrac{b}{x}$

Taking inverse on both sides gives

$$A-AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$$

as $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^{-1})^{-1} = A$.

$$\Rightarrow A = AX + B^{-1}X = (A+B^{-1})X$$

Is $(A+B^{-1})$ invertible? YES, as A and X are invertible

Caution! A, B invertible $\not\Rightarrow$ $A+B$ is invertible.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, but $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\det A = 1$ $\quad \det B = 1$ $\quad \det(A+B) = 0$

$$\left(A = (A+B^{-1})X\right)X^{-1} \Rightarrow \left[AX^{-1} = A+B^{-1}\right]^{-1} \Rightarrow (A+B^{-1})^{-1} = XA^{-1}$$

$$(A+B^{-1})^{-1}\left[A = (A+B^{-1})X\right] \Rightarrow$$

$$(A+B^{-1})^{-1}A = IX = X, \quad \text{i.e., } X = (A+B^{-1})^{-1}A.$$

Inverse of $n \times n$ Matrices

$$[A|I] \xrightarrow{\text{EROs}} [I|A^{-1}] \text{ if } A \text{ is invertible.}$$

Prob 32, pg 127 Find inverse of $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -5 \end{bmatrix}$

modified

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-4R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -3 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 2 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 10 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2-R_3,]{R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & 4 & -1 \\ 0 & 1 & 0 & -14 & 3 & -1 \\ 0 & 0 & 1 & -10 & 2 & -1 \end{array}\right]$$

and $R_3 \times (-1)$

Prob 22, pg 126 Explain why columns of $A \in \mathbb{R}^{n\times n}$ span $\mathbb{R}^n$ when A is invertible.

Since A is invertible, it is row-equivalent to I_n. Hence, it has a pivot in every column and every row. Since there is a pivot in every row, columns of A span $\mathbb{R}^n$.

Prob 33 Pg 127 Find inverses of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Guess the inverse of $A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \vdots \\ \vdots & 1 & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right)^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Guess: $A^{-1} = \begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & -1 & \ddots & \\ 0 & & -1 & 1 \end{bmatrix}$?

$$AA^{-1} = \begin{array}{c} \begin{array}{cccccc} & & j & j+1 & & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & & & \vdots \\ \vdots & \vdots & 1 & & \ddots & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} & j & \end{array} \\ \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & -1 & 1 \end{bmatrix} \end{array}$$

$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th}$ position for $1 \le j \le n-1$

$$(1)\begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow j \\ \\ \\ \\ \end{array} + (-1)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow j+1 \\ \\ \\ \end{array} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_j}$$ the j^{th} unit vector.

So $AA^{-1} = I_n$. Can check that $A^{-1}A = I_n$ as well.

Invertible Matrix Theorem (IMT) (Section 2.3)

Theorem 8, DL-LAA Page 129.

The following statements are equivalent for $A \in \mathbb{R}^{n\times n}$.

a. A is invertible. → A^{-1} is invertible as well.

b. A is row equivalent to I_n.

c. A has n pivots. → pivot in every row & pivot in every column

d. $A\bar{x}=\bar{0}$ has only the trivial solution.

e. Columns of A are LI.

f. The LT $\bar{x} \mapsto A\bar{x}$ is 1-to-1.

g. $A\bar{x}=\bar{b}$ has a unique solution for every $\bar{b} \in \mathbb{R}^n$.

h. Columns of A Span $\mathbb{R}^n$.

i. $\bar{x} \mapsto A\bar{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

j. $\exists\ C \in \mathbb{R}^{n\times n}$ such that $CA=I_n$.

"There exists"

k. $\exists\ D \in \mathbb{R}^{n\times n}$ such that $AD=I_n$

l. A^T is invertible.

Prob 17, Pg 133 If $A \in \mathbb{R}^{n\times n}$ is invertible, then columns of A^{-1} are LI. Explain why.

could skip

Since A is invertible, A^{-1} is invertible. Hence A^{-1} is row equivalent to I_n, i.e, it has a pivot in every column. Hence columns of A^{-1} are LI.

Prob 20, Pg 133 $E, F \in \mathbb{R}^{n\times n}$ satisfy $EF = I$. Explain why E and F commute.

E and F commute means $EF = FE$

Since $E, F \in \mathbb{R}^{n\times n}$ and $EF = I$, both E and F are invertible. Hence, from $EF = I$, we get

$$F = E^{-1}I = E^{-1} \quad \text{and} \quad E = F^{-1}$$

So $EF = FE = I$.

MATH230 – Lecture 18 (03/10/2011)

Prob 28 pg 133

Show that if AB is invertible, and A and B are square, then both A and B are invertible.

there exists

AB is invertible $\Rightarrow \exists\, C \in \mathbb{R}^{n\times n}$ such that

$$\underbrace{CA}B = I \Rightarrow (CA)B = I$$

$\Rightarrow \exists\, E \in \mathbb{R}^{n\times n}$ such that $EB = I$, where $E = CA$. Hence B is invertible.

AB is invertible $\Rightarrow \exists\, D \in \mathbb{R}^{n\times n}$ such that

$$A\underbrace{BD} = I$$

$\Rightarrow \exists\, F \in \mathbb{R}^{n\times n}$ such that $AF = I$, where $F = BD$. $\Rightarrow A$ is invertible.

Prob 36, pg 133

T is an LT which maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Show that T^{-1} exists, and maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Is T^{-1} 1-to-1?

T is an LT mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$.

$\Rightarrow$ $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{n\times n}$, with a pivot in every row. Hence A is invertible.

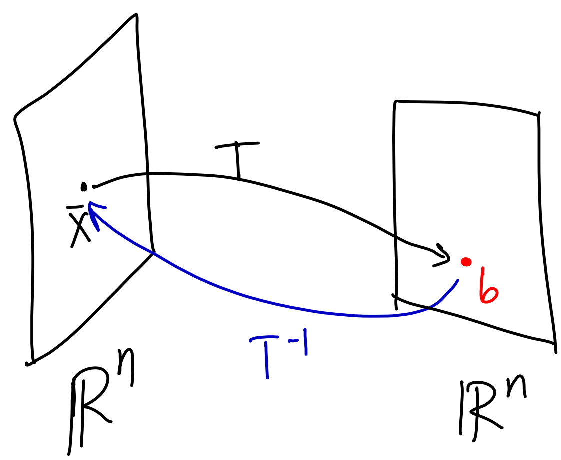
Detour T^{-1} is the inverse LT of T, i.e., if

$$T(\bar{x}) = \bar{b} \text{ then } T^{-1}(\bar{b}) = \bar{x}$$

$T(\bar{x}) = A\bar{x}$ so

$T^{-1}(A\bar{x}) = \bar{x}$. Hence

$T^{-1}(\bar{x}) = A^{-1}\bar{x}$, so that



$T^{-1}(A\bar{x}) = A^{-1}(A\bar{x}) = \bar{x}$

$$[A \mid I] \xrightarrow{\text{EROs}} [I \mid A^{-1}]$$

Since $A \sim I$, $A^{-1} \sim I$. row equivalent.

Hence A^{-1} has a pivot in every row and every column. Hence $T^{-1}(\bar{x})$ is also onto and 1-to-1.

Prob 33 Pg 133

$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x_1, x_2) = (-5x_1 + 9x_2,\ 4x_1 - 7x_2).$

Show that T is invertible and find a formula for T^{-1}!

T is invertible if $T(\bar{x}) = A\bar{x}$ and A is invertible.

Here, $T(\bar{x}) = A\bar{x}$, where $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$

$$A^{-1} = \frac{1}{(-5\times -7 - 4\times 9)} \begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$$

Since A is invertible, T is invertible, and

$$T^{-1}(\bar{x}) = A^{-1}\bar{x}.$$

Hence $T^{-1}(x_1, x_2) = (7x_1 + 9x_2,\ 4x_1 + 5x_2)$

Determinants (Section 3.1)

Recall, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A = ad - bc$. If $\det A \neq 0$,

then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

In general, A is invertible if det $A \neq 0$.

determines if A is invertible.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}$. When is A invertible?

what condition should a, b, c, d satisfy?

We need a pivot in every row and every column.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & b-12 \\ 0 & c-10 & d-15 \end{bmatrix} \xrightarrow{R_3 - \left(\frac{c-10}{a-8}\right)R_2}$$

$a - 8 \neq 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & b-12 \\ 0 & 0 & (d-15) - \dfrac{(c-10)}{(a-8)}(b-12) \end{bmatrix}$$

$\neq 0$ to get three pivots

$$\Rightarrow (a-8)(d-15)-(c-10)(b-12) \neq 0$$

$$ad-8d-15a+\cancel{120}-[bc-10b-12c+\cancel{120}] \neq 0$$

$$(ad-bc)-(8d-10b)+(12c-15a) \neq 0$$

$$\Rightarrow 1(ad-bc)-2(4d-5b)+3(4c-5a) \neq 0$$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}$ We can write,

$$1 \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + (-1)\, 2 \det\left(\begin{bmatrix} 4 & b \\ 5 & d \end{bmatrix}\right)$$

$$+ 3 \det\left(\begin{bmatrix} 4 & a \\ 5 & c \end{bmatrix}\right)$$

This is the expression for $\det A$, expanding along row 1.

In general, for $A \in \mathbb{R}^{n \times n}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n},$$

where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by removing Row 1 and column j from A.

expanding along Row 1.

Prob 8 pg 190

$$\begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix} = 8\begin{vmatrix} 0 & 3 \\ -2 & 5 \end{vmatrix} - 1\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix}$$

$|A| \to$ notation for $\det A$

$$= 8(0\times 5 - -2\times 3) - 1(4\times 5 - 3\times 3) + 6(4\times -2 - 3\times 0)$$

$$= 48 - 11 - 48 = -11.$$

We can expand along any row or any column of A to calculate $\det A$. Let $C_{ij} = (-1)^{i+j}\det A_{ij}$ be the (i,j)-cofactor of A.

Expanding along Column j,

$$\det A = a_{1j}C_{1j} + \dots + a_{nj}C_{nj},$$

and along Row i

$$\det A = a_{i1}C_{i1} + \dots + a_{in}C_{in}.$$

$$\begin{vmatrix} 8 & \boxed{1} & 6 \\ 4 & \boxed{0} & 3 \\ 3 & \boxed{-2} & 5 \end{vmatrix} = (-1)^{1+2}1\cdot\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} + \cancel{(-1)^{2+2}0\cdot\begin{vmatrix} 8 & 6 \\ 3 & 5 \end{vmatrix}} + (-1)^{2+3}2\cdot\begin{vmatrix} 8 & 6 \\ 4 & 3 \end{vmatrix}$$

$$= -11 + 0 + 0 = -11$$

Prob 14, Pg 191

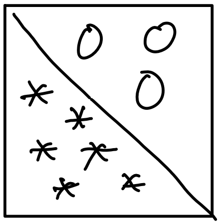
$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = (-1)^{4+1} \cdot 3 \cdot \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{vmatrix}$$

$$= (-3) \cdot (-1)^{3+4} \cdot 1 \cdot \begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

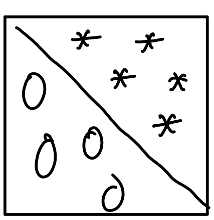
$$= 3\left\{ 3(-4\times 2 - 3\times 1) + 2(2\times 1 - -4\times 4) \right\} = 3(-33+36) = 9.$$

Theorem 2, DL-LAA pg 189

If $A \in \mathbb{R}^{n\times n}$ is triangular, then $\det A$ is the product of the entries in the diagonal.



→ lower triangular (all entries above diagonal are zero).



→ upper triangular

$$A = \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{matrix}} & \begin{matrix} * \\ a_{22} \\ 0 \\ \vdots \end{matrix} & \begin{matrix} * & * \\ * & \\ \ddots & \\ 0 & a_{nn} \end{matrix} \end{bmatrix}$$

$$\det A = a_{11} \cdot \begin{vmatrix} a_{22} & * & * \\ 0 & a_{33} & * \\ \vdots & \ddots & \ddots \\ 0 & 0 & a_{nn} \end{vmatrix}$$

$$= a_{11} \cdot a_{22} \cdot \cdots a_{nn}.$$

Properties of determinants

Example. $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow{R_2 + kR_1} \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix} = A'$

$\det A = 3 \times 6 - 5 \times 4 = -2$

$$\det A' = 3(6+4k) - 4(5+3k)$$

$$= -2 + \cancel{12k} - \cancel{12k}$$

$$= \det A.$$

In general, replacement EROs do not change $\det A$.

MATH 230 – Lecture 19 (03/22/2011)

Office Hours for tomorrow (Wed, Mar 23) are from noon - 2 pm.
Otherwise, I'll be in Neill 216 (come get me from there).

Determinants $A \in \mathbb{R}^{n \times n}$

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$ → expansion along Row i

where $C_{ij} = (-1)^{i+j} \det A_{ij}$ with $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ the submatrix of A obtained by deleting Row i and Column j.

Question: Can we use EROs to simplify determinant calculations?

Theorem 3, DL-LAA pg 192

(a) Replacement EROs do not change the determinant.

If $A \xrightarrow{R_i + kR_j} B$, $\quad (A, B \in \mathbb{R}^{n\times n})$

then $\det A = \det B$.

(b) $A \xrightarrow{R_i \rightleftarrows R_j} B$. Then $\det B = -\det A$.

(c) $A \xrightarrow[k \neq 0]{kR_i} B$. Then $\det B = k \cdot \det A$.

Using EROs to evaluate det A

Let U be the echelon form of A obtained using only replacement and exchange EROs. Then

$$\det A = \begin{cases} (-1)^r \times (\text{product of pivots in } U), & r = \# \text{ exchange EROs} \\ 0 & \text{if } A \text{ is not invertible.} \end{cases}$$

Remember, we **do not use any scaling EROs** to get U from A.

Prob 8 pg 199 Find determinant using row reduction.

$$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} \xrightarrow[R_4+3R_1]{R_3-2R_1} \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{vmatrix} \xrightarrow{R_3+R_2} \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \\ 0 & 2 & 4 & -10 \end{vmatrix} = 0$$

We can also combine row reduction and cofactor expansion to find determinants.

Prob 14 Pg 199

$$\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} \xrightarrow{R_3+2R_1} \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -9 & 0 & 0 & 0 \\ 3 & -4 & 0 & 4 \end{vmatrix}$$

expand along column 3

$$= (-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix} = (-1)^{2+1}(-9)\begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix}$$

$$= 9\left(3\times 4 - (-3)(-4)\right) = 0.$$

Note. $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det A \neq 0$.

Theorem 5 DL-LAA pg 196

$$\det A^T = \det A$$

Proof (By induction).

Show for base case, i.e., $n=1$.
Assume result holds for $n=k$.
Show it holds for $n=k+1$.

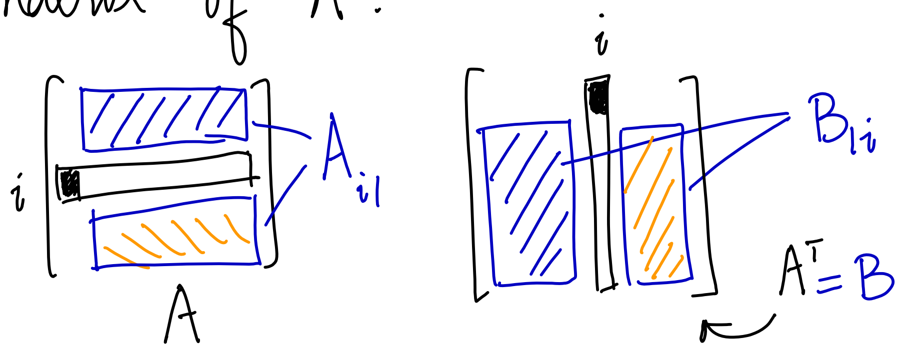
Let $A \in \mathbb{R}^{n \times n}$.

The result holds for $n=1$, as taking transpose of just a number gives back the same number.

Assume result holds for $n=k$, i.e., for any $k \times k$ matrix B, $\det B^T = \det B$.

Consider $A \in \mathbb{R}^{(k+1)\times(k+1)}$, i.e., for $n=k+1$.

Recall that Row i of A is the same as Column i of A^T. Hence the cofactor matrix A_{ij} of A will be the transpose of the corresponding cofactor matrix of A^T. For instance, consider A_{i1}:



$$A_{i1}^T = B_{1i}$$

By induction assumption, the cofactors are same in the expansion for $\det A$ along Row i and $\det A^T$ along column i. Hence the result holds for $n = k+1$, and hence for all n.

Theorem 6 DL-LAA pg 196

$$\det AB = \det A \det B \quad \text{for } A, B \in \mathbb{R}^{n\times n}$$

Warning! $\det(A+B) \neq \det A + \det B$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

det: $1 \qquad 3 \quad \neq \quad 7$

What is $\det A^{-1}$? (Assuming $\det A \neq 0$)

$$\det A^{-1} = \frac{1}{\det A} \quad \text{if } \det A \neq 0.$$

As $AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det I = 1$

$\Rightarrow \det A \cdot \det A^{-1} = 1.$ $\Rightarrow \det A = \dfrac{1}{\det A^{-1}}$.

Prob 32 pg 200

Formula for $\det(rA)$ when $A \in \mathbb{R}^{n\times n}$.

multiply each entry by r.

$$\det(rA) = (r)^n \det A. \qquad A = [a_{ij}]$$

Expanding along Row 1,

$(n-1)\times(n-1)$ matrix

$$\det(rA) = \sum_{j=1}^{n} (-1)^{1+j} (r a_{1j}) \cdot \det(rA_{1j})$$

For each dimension, we get a factor of r multiplying $\det A$ to get $\det rA$. Hence,

$$\det(rA) = (r)^n \det A.$$

Prob 40 pg 200

$A, B \in \mathbb{R}^{4\times 4}$ with $\det A = -1$, $\det B = 2$.

(a) $\det AB = \det A \cdot \det B = -1 \times 2 = -2$

(b) $\det B^5 = (\det B)^5 = 2^5 = 32$

(e) $\det B^{-1}AB = \cancel{\det B} \cdot \det A \cdot \frac{1}{\cancel{\det B}} = -1.$

Computer Project - Write your own rref function

Row reduction algorithm $\longrightarrow$ a step-by-step procedure to row reduce any $m \times n$ matrix to its echelon form.

Iteration $\longrightarrow$ repeat a step or calculation for several indices (or for several rows or columns).

e.g., Scale Row 1 of $A \in \mathbb{R}^{m \times n}$ by k.

With $A = [a_{ij}]$, we want $a_{11} \longrightarrow k a_{11}$, $a_{12} \longrightarrow k a_{12}, \ldots,$
$a_{1n} \longrightarrow k a_{1n}$.

"replace"

$$\left.\begin{array}{l} \textbf{for } j = 1 \text{ to } n \\ \quad a_{1j} \longrightarrow k a_{1j} \\ \textbf{end} \end{array}\right\} \text{a "for" loop}$$

MATH 230 – Lecture 20 (03/24/2011)

Iterations in MATLAB

e.g., adding 2 vectors. $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. To find $\bar{c} = \bar{a} + \bar{b}$, we can write a for loop.

for $i = 1$ to n
  $\bar{c}(i) = \bar{a}(i) + \bar{b}(i)$
end

To square every entry in $A \in \mathbb{R}^{m \times n}$.

With $A = [a_{ij}]$, we want $a_{ij} \leftarrow a_{ij}^2$

replace.

for $i = 1$ to m for each row
  for $j = 1$ to n
    $a_{ij} \leftarrow a_{ij}^2$
  end
end

doing it row-wise

Could also do

for $j = 1$ to n for each column
  for $i = 1$ to m
    $a_{ij} \leftarrow a_{ij}^2$
  end
end

column-wise addition

Functions in MATLAB

A "black-box" which takes some input and gives some output. For instance, we can write a function that takes as input two vectors $\bar{a}, \bar{b}$ and outputs their sum $\bar{c}$, i.e., $\bar{c} = \bar{a} + \bar{b}$.

output

```
function  c̄ = Sum(ā, b̄)

    n = length(ā);
    c̄ = zeros(n,1);

    for i = 1 to n
        c(i) ← a(i) + b(i);
    end
```

inputs assuming $\bar{a}, \bar{b} \in \mathbb{R}^{n \times 1}$

Column vectors.

Save all the commands as the file Sum.m

The word sum is a keyword in MATLAB already. You do not want reuse preset keywords. MATLAB usually complains if you try to do so.

See MATLAB output on course webpage!

MATH 230 – Lecture 21 (03/29/2011)

Today's office hours – 2:45-4:00 pm and then from 5:00-6:00 pm.

Vector Spaces (Section 4.1)

Def: A real vector space is a non-empty set of objects V, on which two operations addition and scalar multiplication are defined, and the objects satisfy the following ten axioms. Here, $\bar{u}, \bar{v}, \bar{w} \in V$, and $c, d \in \mathbb{R}$.

$\in$ – element of

1. $\bar{u}+\bar{v} \in V$.
2. $\bar{u}+\bar{v} = \bar{v}+\bar{u}$.
3. $(\bar{u}+\bar{v})+\bar{w} = \bar{u}+(\bar{v}+\bar{w})$.
4. $\exists\ \bar{0} \in V$, the **zero** of V, such that $\bar{u}+\bar{0} = \bar{0}+\bar{u} = \bar{u}$.

   $\exists$ – There exists

5. $\forall\ \bar{u} \in V$, $\exists$ an object $-\bar{u} \in V$ such that $\bar{u}+-\bar{u}=\bar{0}$

   $-\bar{u}$ is the **inverse** of $\bar{u}$.

6. $c\bar{u} \in V$.

7. $c(\bar{u}+\bar{v}) = c\bar{u} + c\bar{v}$.

8. $(c+d)\bar{u} = c\bar{u} + d\bar{u}$.

9. $c(d\bar{u}) = (cd)\bar{u}$

10. $1\bar{u} = \bar{u}$.

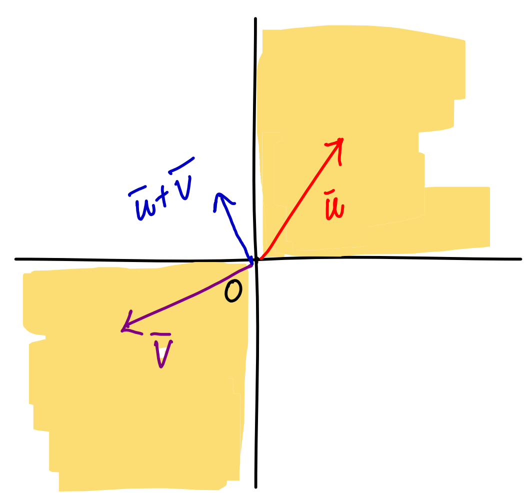
The main axioms are 1, 4, 6, i.e., V should have a zero, it should be closed under and under scalar multiplication.

e.g., $\mathbb{R}^2$, $\mathbb{R}^n$, $\{\bar{0}\}$ are vector spaces.
$\{\bar{0}\} \to$ single element set

Prob 2 pg 223

W is the union of first and third quadrants in $\mathbb{R}^2$.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; xy \geq 0 \right\}$$



(a) If $\bar{u} \in W$, $c \in \mathbb{R}$, is $c\bar{u} \in W$?

$\bar{u} \in W \Rightarrow \bar{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $xy \geq 0$

$c\bar{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}$. Hence the product $(cx)(cy) = c^2 xy \geq 0$ $\forall c \in \mathbb{R}$.

(b) But W is not closed under addition. Let

$\bar{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$. Both $\bar{u}, \bar{v} \in W$. But

$\bar{u} + \bar{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin W$.

Hence W is not a vector space.

For $n \geq 0$, $\mathbb{P}_n$ is the collection of all polynomials of degree up to n (including n).

$p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$ is an n^{th} degree polynomial if $a_n \neq 0$.

Note: $p(x) = a_0 + \dots + a_n x^n$ is the same as $p(t)$.

Claim: $\mathbb{P}_n$ is a vector space.

Def: $p(t) = a_0 \neq 0$ is a zero-degree polynomial

$p(t) = 0$ is the **zero polynomial** $\in \mathbb{P}_n$.

But degree of the zero polynomial is undefined.

Let $q(t) = b_0 + b_1 t + \cdots + b_n t^n$

Then $p(t) + q(t)$ is the sum of the two polynomials.

$$(p+q)(t) = p(t) + q(t) = (a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_n t^n)$$

$$= (a_0 + b_0) + (a_1 + b_1) t + \cdots + (a_n + b_n) t$$

$$= c_0 + c_1 t + \cdots + c_n t^n \in \mathbb{P}_n.$$

Similarly, $c\,p(t)$ for $c \in \mathbb{R}$ is

$$c(a_0 + \cdots + a_n t^n) = c a_0 + \cdots + c a_n t^n \in \mathbb{P}_n.$$

Hence $\mathbb{P}_n$ is a vector space.

MATH 230 – Lecture 22 (03/31/2011)

$\mathbb{P}_n$ is a vector space (continued...)

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n, \quad a_j \in \mathbb{R} \text{ for } j = 0, \ldots, n$$

For any $c \in \mathbb{R}$, $c\,p(t) = c(a_0 + \cdots + a_n t^n)$

$$= ca_0 + \cdots + ca_n t^n$$

$$= c_0 + \cdots + c_n t^n = q(t)$$

Here $c_j = c\,a_j$; hence $q(t) \in \mathbb{P}_n$.

All other axioms are satisfied by polynomials. For instance, $-p(t) = -(a_0 + \cdots + a_n t^n)$ is such that $p(t) + -p(t) = 0$, the zero polynomial.

Prob 25, Page 224

Show that the zero of a vector space is unique.

Def. $\bar{0}$ is an element of V such that $\forall\, \bar{u} \in V$ "for all"

$$\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}.$$

Suppose there is a $\bar{w} \in V$ such that $\forall \bar{u} \in V$, ($\bar{w}$ is different from $\bar{0}$)

$$\bar{w} + \bar{u} = \bar{u} + \bar{w} = \bar{u}.$$

This result holds for $\bar{0} \in V$ as well. Hence

$$\bar{0} + \bar{w} = \bar{0} \implies \bar{w} = \bar{0},$$ i.e, the zero of V

is unique (as $\bar{w}$ is not another zero, it's just identical to $\bar{0}$).

proof by contradiction. → $\bar{w} = \bar{0}$, which contradicts our starting assumption that they are different.

Subspaces

~ "a subset of V, which is a vector space"

Def. A subspace of a vector space V is a subset H of V $(H \subseteq V)$ such that

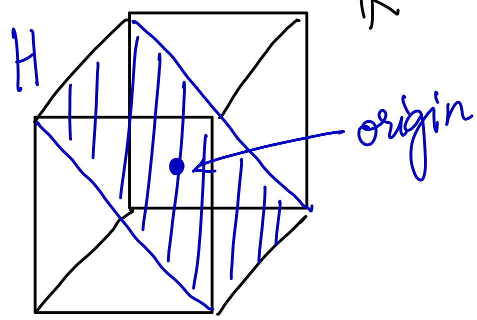
(subset or equal to)

(a) $\bar{0} \in H$, where $\bar{0}$ is the zero of V;

(b) $\forall\, \bar{u}, \bar{v} \in H$, $\bar{u} + \bar{v} \in H$, i.e., H is closed under addition; and

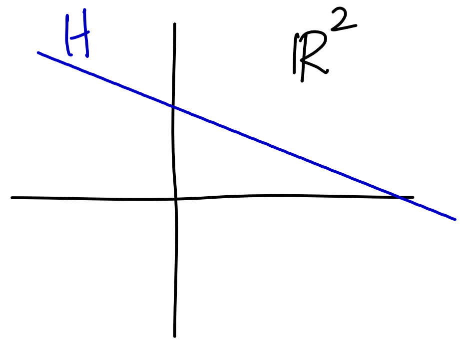
(c) $\forall\, \bar{u} \in H$, $c \in \mathbb{R}$, $c\bar{u} \in H$, i.e., H is closed under scalar multiplication.

Examples



H is a plane in $\mathbb{R}^3$ passing through the origin.

H is a subspace of $\mathbb{R}^3$.



H is a line not passing through the origin. Then H is not a subspace of $\mathbb{R}^2$.

Prob 13 pg 223

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \text{ and } \bar{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

(a) Is $\bar{w}$ in $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$? How many vectors are there in $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

No. $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is just the collection of the vectors $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$.

For any set S, $|S|$ denotes the # elements in S.

$|\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}| = 3.$

notation for size of a set, i.e, # entries in the set.

(b) Is $\bar{w}$ in $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$?

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array}\right] \xrightarrow{R_3-5R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has many solutions. Hence $\bar{w} \in \text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$.

(c) How many vectors are there in $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$?

$|\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)| = \infty$, as there are infinitely many linear combinations of $\bar{v}_1, \bar{v}_2, \bar{v}_3$.

<u>Prob 7, pg 223</u> Let W be the set of all polynomials of degree at most 3, with integer coefficients. Is W a subspace of $\mathbb{P}_n$ for $n \geq 3$?

collection of all polynomials with degree up to n.

$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, $a_j \in \mathbb{Z}$
$j = 0, 1, 2, 3$

set of all integers

W is **not closed under scalar multiplication**, as $ca_j \notin \mathbb{Z}$ when c is a fraction. $\bar{0} \in W$, and W is closed under addition. Hence W is not a subspace of $\mathbb{P}_n$, $n \geq 3$.

e.g., $p(t) = 1 + 2t \in W$, but $\frac{1}{3}p(t) = \frac{1}{3} + \frac{2}{3}t \notin W$.

Prob 8 pg 233

W is the set of all polynomials $p(t)$ in $\mathbb{P}_n$ such that $p(0) = 0$. Is W a subspace of $\mathbb{P}_n$?

$0 \in W$, as $p(0) = 0$ is present in W.

W closed under addition? $p(0) + q(0) = 0 + 0 = 0$.

YES.

W closed under scalar multiplication? $c\,p(0) = c \cdot 0 = 0$.

YES.

So, W is a subspace of $\mathbb{P}_n$.

need not be vectors

Theorem 1, DL-LAA pg 221 If $\bar{v}_1, \ldots, \bar{v}_p \in V$, a vector space, then $H = \text{span}(\bar{v}_1, \ldots, \bar{v}_p)$ is a subspace of V.

Proof $\bar{0} \in H$, as $\text{span}(\bar{v}_1, \ldots, \bar{v}_p) = \left\{ \sum_{j=1}^{p} c_j \bar{v}_j \mid c_j \in \mathbb{R} \right\}$.

Taking $c_j = 0$ for all j gives $\bar{0}$, the zero of V.

H is closed under addition. For $\bar{u}, \bar{w} \in H$,

$$\bar{u} = \sum_{j=1}^{p} c_j \bar{v}_j, \quad \bar{w} = \sum_{j=1}^{p} d_j \bar{v}_j, \text{ we get}$$

$$\bar{u} + \bar{w} = \sum_{j=1}^{p} (c_j + d_j) \bar{v}_j \in H$$

H is closed under scalar multiplication. For $\bar{u} \in H$, $d \in \mathbb{R}$ with $\bar{u} = \sum_{j=1}^{p} c_j \bar{v}_j$, $d\bar{u} = \sum_{j=1}^{p} (dc_j) \bar{v}_j \in H$.

<u>Prob 17, pg 223</u>

"such that"

$$W = \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}.$$ Is W a subspace of $\mathbb{R}^4$?

If yes, find a set of vectors S, such that $W = \text{span}(S)$.

$\bar{0} \in W$; take $a = b = c = 0$.

⋮

But, we could rather just find S such that $W = \text{span}(S)$.

This result would confirm that W is a subspace

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\bar{v}_1} + b\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_2} + c\underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_3} \; ; \quad a, b, c \in \mathbb{R}$$

Hence $W = \text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$, where $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

Hence, W is a subspace of $\mathbb{R}^4$.

<u>Prob 16, pg 223</u>

$W = \left\{ \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}, \; a, b \in \mathbb{R} \right\}$. Is W a subspace of $\mathbb{R}^3$.

No, as $\bar{0} \notin W$. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$ has no solutions a, b.

$\left.\begin{matrix} a-6b=0 \\ 2b+a=0 \end{matrix}\right\}$ gives $a=b=0$ as the unique solution, which does not agree with $-a+1=0$, which needs $a=1$.

MATH 230 – Lecture 23 (04/05/2011)

Office hours today: 2:45–4:00 pm and 5–6:00 pm.

Recall H is a subspace of a vector space V, if

(i) $\bar{0} \in H$,

(ii) $\forall\, \bar{u}, \bar{v} \in H$, $\bar{u} + \bar{v} \in H$,

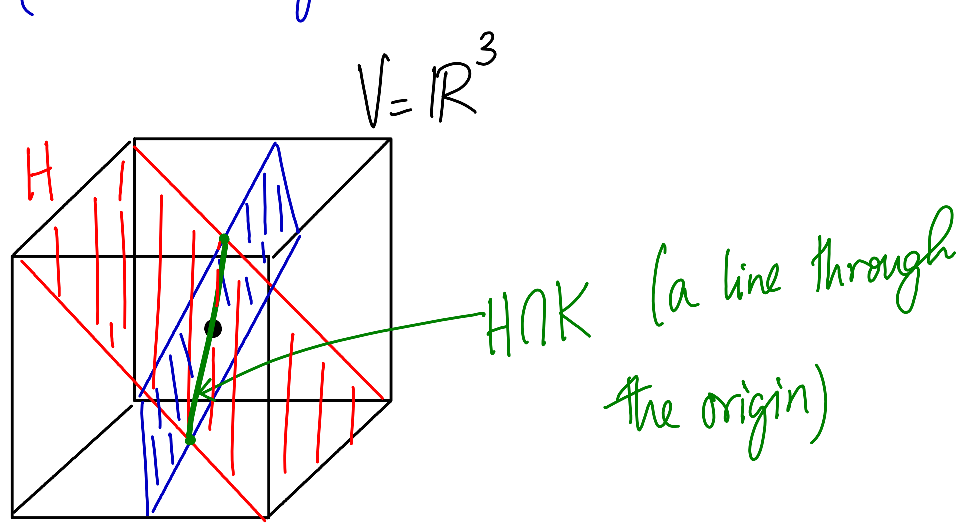
(iii) $\forall\, \bar{u} \in H$, $c \in \mathbb{R}$, $c\bar{u} \in H$.

Prob 32 pg 225

Let H and K be subspaces of a vector space V. The set $H \cap K$ is the collection of all elements of V that are in both H and K. Show that $H \cap K$ is a subspace of V. (intersection of subspaces is also a subspace).

intersect

Example:



Proof We show that $H \cap K$ satisfies the three axioms for being a subspace, i.e., it includes zero, and is closed under addition and scalar multiplication.

H and K are subspaces of V.

$$\Rightarrow \left.\begin{array}{l} \bar{0} \in H \\ \bar{0} \in K \end{array}\right\} \Rightarrow \bar{0} \in H \cap K.$$

Consider $\bar{u}, \bar{v} \in H \cap K$. Hence $\bar{u}, \bar{v} \in H$ and $\bar{u}, \bar{v} \in K$.

$$\left.\begin{array}{l} \forall\, \bar{u}, \bar{v} \in H,\ \bar{u}+\bar{v} \in H \\ \forall\, \bar{u}, \bar{v} \in K,\ \bar{u}+\bar{v} \in K \end{array}\right\} \Rightarrow \forall\, \bar{u}, \bar{v} \in H \cap K,\ \bar{u}+\bar{v} \in H \cap K.$$

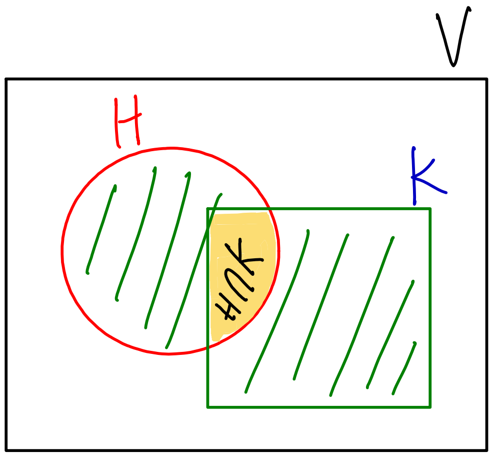
Consider $\bar{u} \in H \cap K$, $c \in \mathbb{R}$. So, $\bar{u} \in H$, and $\bar{u} \in K$.

$$\left.\begin{array}{l} \forall\, \bar{u} \in H,\ c \in \mathbb{R},\ c\bar{u} \in H \\ \forall\, \bar{u} \in K,\ c \in \mathbb{R},\ c\bar{u} \in K \end{array}\right\} \Rightarrow \forall\, \bar{u} \in H \cap K,\ c \in \mathbb{R},\ c\bar{u} \in H \cap K.$$

Hence $H \cap K$ is a subspace of V.

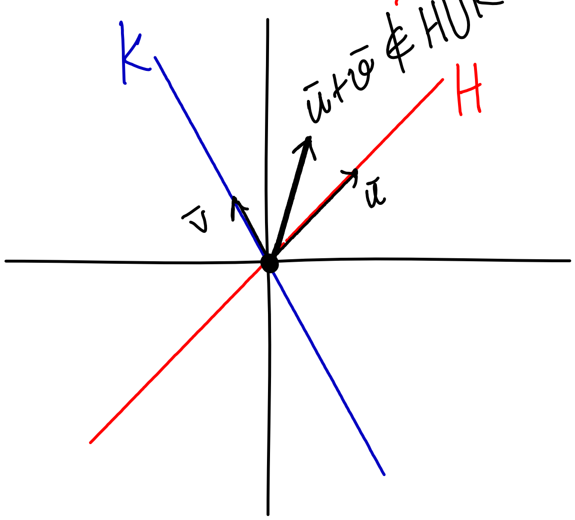
What about $H \cup K$?

"union" → collection of all entries in V that are present in either H or K, but not necessarily in both.



$/// \to H \cup K$

A Venn diagram illustrating union and intersection of sets.



"not element of" or "not in"

The union of two lines, which are both subspaces, is not closed under addition.

As illustrated by this example in $\mathbb{R}^2$, the union of two subspaces is typically not a subspace.

Null Space and Column Space of A (Section 4.2)

$A \in \mathbb{R}^{m\times n}$

Def The **null space** of A, denoted by $\operatorname{Nul} A$ or $\operatorname{Nul}(A)$, is the set of all solutions to $A\bar{x}=\bar{0}$.

$$\operatorname{Nul} A = \{\bar{x}\in\mathbb{R}^n \mid A\bar{x}=\bar{0}\}$$

Theorem 2, DL-LAA pg 227 $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

Proof (i) $\bar{0}\in \operatorname{Nul} A$, as $A\bar{0}=\bar{0}$. ($\bar{0}$ is the trivial solution to $A\bar{x}=\bar{0}$.)

(ii) Let $\bar{u},\bar{v}\in \operatorname{Nul} A$. Then $A\bar{u}=0$, $A\bar{v}=\bar{0}$.

$\Rightarrow A\bar{u}+A\bar{v} = A(\bar{u}+\bar{v}) = \bar{0}$, i.e., $\bar{u}+\bar{v}\in \operatorname{Nul} A$.

(iii) Let $\bar{u}\in \operatorname{Nul} A$, $c\in\mathbb{R}$. Then $A\bar{u}=\bar{0}$.

$\Rightarrow cA\bar{u} = A(c\bar{u}) = \bar{0}$, i.e., $c\bar{u}\in \operatorname{Nul} A$.

Hence, $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

An explicit description of $\operatorname{Nul} A$ is obtained by finding the parametric-vector form of all solutions to $A\bar{x}=\bar{0}$.

Prob 4, Pg 234 Find an explicit description of Nul A by listing a set of vectors that span Nul A, where

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \xrightarrow[\text{then } \frac{1}{2}R_2]{R_1 - 2R_2} \begin{array}{c} \quad x_2 \qquad\quad x_4 \text{ free} \\ \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$x_1 = 6x_2$
$x_3 = 0$, $x_2, x_4 \in \mathbb{R}$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4, \quad x_2, x_4 \in \mathbb{R}.$$

Hence $\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, which is a subspace of $\mathbb{R}^4$.

Prob 8 Pg 234

$W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} \middle| \; 5r - 1 = s + 2t \right\}$. Is W a subspace of $\mathbb{R}^3$?

W is not a subspace, as $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$.

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not satisfy $5r - 1 = s + 2t$.

Prob 9, Pg 234

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; \begin{matrix} a-2b=4c \\ 2a=c+3d \end{matrix} \right\}.$$ Is W a subspace of $\mathbb{R}^4$?

$$\left.\begin{matrix} a-2b=4c \\ 2a=c+3d \end{matrix}\right\} \Longrightarrow \left.\begin{matrix} a-2b-4c = 0 \\ 2a - c - 3d = 0 \end{matrix}\right\} \Longrightarrow$$

$$A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \bar{0} \quad \text{for } A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}.$$

Hence $W = \text{Nul } A$, and hence it is a subspace of $\mathbb{R}^4$.

Prob 1 pg 234

$\bar{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$. Is $\bar{w} \in \text{Nul } A$?

$A\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\bar{w} \in \text{Nul } A$.

As illustrated here, it is relatively easy to check if $\bar{w} \in \text{Nul } A$. We just calculate $A\bar{w}$.

Column Space of A

The **column space** of A, denoted by $\text{Col } A$ or $\text{Col}(A)$, is the set of all linear combinations of the columns of A.

$$\text{Col}\,A = \{\bar{b} \in \mathbb{R}^m \mid A\bar{x} = \bar{b} \text{ for some } \bar{x} \in \mathbb{R}^n\}$$

→ set of all right hand-sides for which $A\bar{x} = \bar{b}$ is consistent.

If $A = [\bar{a}_1\, \bar{a}_2 \cdots \bar{a}_n]$, then

$$\text{Col}\,A = \left\{ \sum_{j=1}^{n} \bar{a}_j x_j \;\middle|\; x_j \in \mathbb{R}\ \forall j \right\}$$

Also, $\text{Col}\,A$ = range of T, where $T(\bar{x}) = A\bar{x}$, i.e., the set of all images of T.

$$\text{Col}\,A = \text{Span}(\bar{a}_1, \ldots, \bar{a}_n).$$

→ hence $\text{Col}\,A$ is a subspace of $\mathbb{R}^m$!

Prob 15, pg 234

$$W = \left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} \;\middle|\; r,s,t \in \mathbb{R} \right\}.$$

Find A such that $W = \text{Col}\,A$.

$$\left.\begin{matrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{matrix}\right\} \overset{\text{"equivalent"}}{\equiv} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} r + \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} s + \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} t = A\bar{x}, \text{ where}$$

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \text{ and } \bar{x} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}. \quad \text{Hence } W = \text{Col}\,A.$$

Since $\text{Col}\,A = \text{span}(\bar{a}_1, \dots, \bar{a}_n)$, $\text{Col}\,A$ is a subspace of $\mathbb{R}^m$ (as each $\bar{a}_j \in \mathbb{R}^m$).

<u>Prob 23 pg 235</u> $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, $\bar{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Is $\bar{w} \in \text{Col}\,A$?

Solve $A\bar{x} = \bar{w}$

$$\left[\begin{array}{cc|c} -6 & 12 & 2 \\ -3 & 6 & 1 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & 6 & 1 \end{array}\right]$$

system is consistent. Hence $\bar{w} \in \text{Col}\,A$.

Unlike in the case of checking whether $\bar{w} \in \text{Nul}\,A$, to check if $\bar{w} \in \text{Col}\,A$, we typically have to do more work — solve $A\bar{x} = \bar{w}$.

But in the above example, one could notice that $A = [\bar{a}_1\ \bar{a}_2]$, where $\bar{a}_1 = -3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\bar{a}_2 = 6\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Hence, $\text{Col}\,A = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$, and as such, $\bar{w} \in \text{Col}\,A$.

We could not make such easy observations on most larger examples, though.

MATH 230 – Lecture 24 (04/07/2011)

Col A and Nul A of $A \in \mathbb{R}^{m\times n}$

Prob 24 pg 235

$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$ Is $\bar{w}$ in Col A? Is $\bar{w}$ in Nul A?

$\bar{w} \in \text{Col}A$ if $A\bar{x} = \bar{w}$ is consistent.

$$\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \xrightarrow[R_2 - \frac{3}{2}R_3]{R_1 + 2R_3} \left[\begin{array}{ccc|c} 0 & -2 & -1 & -2 \\ 0 & 4 & 2 & 4 \\ 4 & 0 & 4 & -2 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|c} 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & -2 \end{array}\right]$$

System is consistent. Hence $\bar{w} \in \text{Col}\,A$.

$$A\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$ so $\bar{w} \in \text{Nul}\,A$.

#from OL-LAA pg232

Nul A v/s	Col A $\quad A \in \mathbb{R}^{m\times n}$
1. Subspace of $\mathbb{R}^n$	1. Subspace of $\mathbb{R}^m$
5. $\bar{w} \in \text{Nul}A$ if $A\bar{w} = \bar{0}$	2. $\bar{w} \in \text{Col}A$ if $A\bar{x} = \bar{w}$ is consistent
7. $\text{Nul}\,A = \{\bar{0}\}$ if there is a pivot in every column	7. $\text{Col}\,A = \mathbb{R}^m$ if A has a pivot in every row.

Prob 27, pg 235

$$x_1 - 3x_2 - 3x_3 = 0$$
$$-2x_1 + 4x_2 + 2x_3 = 0$$
$$-x_1 + 5x_2 + 7x_3 = 0$$

$\bar{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is a solution for the given system. Explain why $\bar{\mathbf{x}}' = \begin{bmatrix} 30 \\ 20 \\ -10 \end{bmatrix}$ is also a solution.

The system is $A\bar{x} = \bar{0}$, for $A = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 2 \\ -1 & 5 & 7 \end{bmatrix}$. $\bar{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is a solution $\Longrightarrow$ $\bar{\mathbf{x}} \in \text{Nul } A$. Hence $c\bar{\mathbf{x}} \in \text{Nul } A$ for all $c \in \mathbb{R}$ (as Nul A is closed under scalar multiplication).

Nul A is a subspace

So $\bar{\mathbf{x}}' = 10\bar{\mathbf{x}}$ is also a solution to $A\bar{x} = \bar{0}$.

Linearly Independent Set and Bases (Section 4.3)

plural for Basis

Recall $\{\bar{v}_1, \ldots, \bar{v}_n\}$ with $\bar{v}_j \in \mathbb{R}^m$ is LI if $n \leq m$ and $A\bar{x} = \bar{0}$ with $A = [\bar{v}_1 \cdots \bar{v}_n]$ has only the trivial solution (A has a pivot in every column).

Def Let H be a subspace of a vector space V. $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_p\}$ with $\bar{b}_j \in V$ for all j, is a **basis** of H if

(i) $\mathcal{B}$ is a linearly independent (LI) set, and

(ii) $\text{span}(\bar{b}_1, \ldots, \bar{b}_p) = H$.

A set $\{\bar{v}_1, \ldots, \bar{v}_p\}$ (not necessarily of vectors), is linearly dependent (LD) if $\bar{v}_j = \sum_{i \neq j} c_i \bar{v}_i$ for some j, i.e., one object is a linear combination of the other objects.

$c_i \in \mathbb{R}$

A set that is not LD is linearly independent (LI).

e.g., colors $\{\overset{R}{\text{Red}}, \overset{G}{\text{Green}}, \overset{B}{\text{Blue}}\}$ is LI, but $\{R, G, B, \text{Purple}\}$ is LD as we can combine Red and Blue to get Purple.

Examples of Bases

① $\{\bar{e}_1, \dots, \bar{e}_n\}$ (the set of unit vectors) is a basis for $\mathbb{R}^n$.

unit vectors

The standard basis of $\mathbb{R}^n$.

② $\{1, t, t^2, \dots, t^n\}$ is a basis for $\mathbb{P}_n$.

$$p(t) = a_0 + a_1 t + \dots + a_n t^n = a_0(1) + a_1(t) + \dots + a_n(t^n),$$

$a_0, a_1, \dots, a_n \in \mathbb{R}$.

standard basis for $\mathbb{P}_n$

③ $\{1, t, 2t^2\}$ is a basis for $\mathbb{P}_2$.

$\{-2, 4t+6, \sqrt{2}t^2\}$ is also a basis for $\mathbb{P}_2$.

Note: This set is indeed LI, as we cannot write any one element as a linear combination of the others — the degrees do not match.

Also, every polynomial $p(t)$ with degree ≤ 2 can be written as $p(t) = a_0(-2) + a_1(4t+6) + a_2(\sqrt{2}t^2)$ for $a_0, a_1, a_2 \in \mathbb{R}$.

e.g., if $p(t) = 2+t$, we need $a_2 = 0$, $4a_1 = 1$, and $6a_1 - 2a_0 = 2$, which give $a_0 = -\frac{1}{4}$, $a_1 = \frac{1}{4}$, $a_2 = 0$.

Bases for Col A and Nul A

Theorem 6 DL-LAA pg 241 The pivot columns of A form a basis for Col A.

the columns in original matrix A, not in its echelon form.

Check: (i) Pivot columns are LI.

(ii) Col A = span (pivot columns)

as non-pivot columns can be written as linear combinations of pivot columns.

Basis for Nul A $\rightsquigarrow$ from parametric vector form of solutions to $A\bar{x} = \bar{0}$.

Prob 13 Pg 243

pivot columns

row equivalent

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad B \sim A.$$

Find bases for Col A and Nul A.

Since columns 1 and 2 are pivot columns,

$\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$ is a basis for Col A.

Need rref(A) for basis of Nul A.

$$B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \times 1/2} \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ x_3, x_4 free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} x_4,$$ $x_3, x_4 \in \mathbb{R}$ are all solutions to $A\bar{x} = \bar{0}$

Hence $\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

Def The number of elements in a basis for subspace H is called its **dimension**.

Result Each basis of H has the same number of elements.

e.g., 1. $\mathbb{R}^2$ is 2-dimensional. (dimension of $\mathbb{R}^2$ is 2).

e.g., $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-1\\4\end{bmatrix}\right\}$ are bases.

2. $\mathbb{P}_2$ is 3-dimensional.

e.g., $\underbrace{\{1, t, t^2\}}_{\text{3 elements.}}$ is a basis.

MATH230 – Lecture 25 (04/12/2011)

Computer project: Illustration

Say we have $A = \begin{bmatrix} 0 & 5 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix}$.

$[m,n] = \text{size}(A)$ gives $m=3$, $n=4$.

$n_e = 0$ the # EROs (is zero at start)

Iteration 1 of main while loop

$i=1, j=1$ start at top-left corner

$$\begin{array}{cc} & j \\ \begin{array}{r} i_1 = i \rightarrow \\ i_1 = 2 \rightarrow \\ i_1 = 3 \rightarrow \end{array} & \begin{bmatrix} 0 & 5 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix} \end{array}$$

$i_1 = 1$, $A(1,1) = 0 \Rightarrow$ go to next row: $i_1 \rightarrow 2$ $(1+1)$

$i_1 = 2$ $A(2,1) = 0 \Rightarrow$ go to next row: $i_1 \rightarrow 3$ $(2+1)$

$i_1 = 3$ $A(3,1) = 2 \neq 0$ pivot found!

$i_{NZ} = i_1 = 3$; nzfound$=1$;

Store the index of the pivot row

so, stop the "while nzfound==0…" loop when you try to repeat it next time

In pseudocode, == is checking for equality, while = is used for assignment.

$i = i+1$ → take current value of i, add 1, and store the result in i itself.

```
if j==5
    DoSomething;
end
```

→ DoSomething is run if j is 5.

The same syntax is used in MATLAB — == for comparison and = for assignment.

---

nzfound $=1$, $i_{nz}=3$, $i=1$, $j=1$ now.

$i_{nz} \neq i$ (pivot row is below current row), so swap $R_{i_{nz}}$ and R_i ($R_3 \rightleftharpoons R_1$ here)

If the pivot were located in Row i itself, we would not have to do a row swap here. The goal is to get the pivot located in Position (i,j) itself, assuming there exists a pivot.

$$A = \begin{bmatrix} 0 & 5 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \rightleftharpoons R_3} \begin{array}{cc} & j \\ i \rightarrow & \left[\begin{array}{cccc} 2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 4 \end{array}\right] \end{array}$$

$n_e = 0+1 = 1$

increment #EROs by 1.

$$R_i \times \frac{1}{A(i,j)} = R_1 \times \frac{1}{A(1,1)} = R_1 \times \frac{1}{2}$$

Scale the pivot so that it becomes 1.

$$A = \begin{bmatrix} 1 & 3/2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 4 \end{bmatrix}$$

$n_e = 1+1 = 2;$

Here, $A(2,1)=0$, $A(3,1)=0$ already. So we do not any replacement EROs to make this pivot column $(j=1)$ a unit vector.

$i=i+1 \Rightarrow i=1+1=2.$ → go to next row

$j=j+1 \Rightarrow j=1+1=2$ → go to next column.

END of Iteration 1 here.

Iteration 2

$$\begin{array}{c} \\ i\rightarrow \\ \\ \end{array}\begin{array}{c}\begin{array}{cccc} & j & & \end{array}\\ \begin{bmatrix} 1 & 3/2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 4 \end{bmatrix}\end{array}$$

We repeat the same set of calculations on the smaller 2×3 matrix.

$(i_1=3)$

$j=2$ now

Start with $i_1=i=2$, $nzfound=0$. → Look for a pivot in Column $j=2$, in Row $i=2$ or lower.

$A(2,2)=0.$ So, $i_1=i_1+1=3.$ → go to next row.

$A(3,2)=5\neq 0.$ $nzfound=1;$ $i_{nz}=i_1=3$ → save pivot row

Now we have $nzfound==1$, $i_{nz}=3$, $i=2$.

Swap R_2 and R_3 $(R_{i_{nz}}$ and $R_i)$ $R_2 \rightleftarrows R_3$ → to get pivot in $A(i,j)$.

$$\begin{array}{c} \\ i\rightarrow \\ \\ \end{array}\begin{array}{c}\begin{array}{cccc} & j & & \end{array}\\ \begin{bmatrix} 1 & 3/2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 4 \end{bmatrix}\end{array} \xrightarrow{R_2 \rightleftarrows R_3} \begin{bmatrix} 1 & 3/2 & 0 & 1 \\ 0 & 5 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n_e=2+1=3$

$$R_i \times \frac{1}{A(i,j)} = R_2\times\frac{1}{5} \longrightarrow \begin{bmatrix} 1 & 3/2 & 0 & 1 \\ 0 & 1 & 2/5 & 4/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n_e=3+1=4$

Scale to make pivot $=1$.

$i=2$ now, $(j=2)$. → zero out non-pivot entries in column j

for $i_1 = 1,2,3$ but $i_1 \neq i$, i.e., for $i_1 = 1, 3$, zero out $A(i_1,j)$.

$i_1 = 1 \Rightarrow A(i_1,j) = A(1,j) = \frac{3}{2}$. So, do $R_1 - \frac{3}{2}R_2$.

In general, do $R_{i_1} - A(i_1,j) \times R_i$ → current pivot row

$$\begin{bmatrix} 1 & 3/2 & 0 & 1 \\ 0 & 1 & 2/5 & 4/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - \frac{3}{2}R_2} \begin{array}{c c} & \begin{array}{cccc} & j & & \end{array} \\ \begin{array}{c} \\ i \rightarrow \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & -3/5 & -1/5 \\ 0 & 1 & 2/5 & 4/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad n_e = 4+1 = 5$$

for $i_1 = 3$, $A(i_1,j) = A(3,2) = 0$, so no replacement ERO is needed.

$i = i+1 = 3$ → go to next row

$j = j+1 = 3$. → go to next column

END of Iteration 2

Iteration 3

$$\begin{array}{c c} & \begin{array}{cccc} & & j & \end{array} \\ \begin{array}{c} \\ \\ i \rightarrow \end{array} & \begin{bmatrix} 1 & 0 & -3/5 & -1/5 \\ 0 & 1 & 2/5 & 4/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

→ Notice that we already have the RREF of A. But the function runs through all rows and all columns before it finishes.

Start with $i_1 = i = 3$, nzfound $= 0$.

$A(i_1,j) = A(3,3) = 0$. nzfound $= 0$ still. → nothing more done.

END of Iteration 3.

Iteration 4

$i = 3$, → since nzfound $= 0$, i is not incremented here.

$j = j+1 = 4$ → go to next column.

$$\begin{array}{c c} & \begin{array}{cccc} & & & j \end{array} \\ \begin{array}{c} \\ \\ i \end{array} & \begin{bmatrix} 1 & 0 & -3/5 & -1/5 \\ 0 & 1 & 2/5 & 4/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

→ nothing more done.

END of RREF!

MATH230 – Lecture 26 (04/14/2011)

Dimension of a vector space = # elements in any basis

Prob 19, pg 244

$\bar{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, $H = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.

It can be shown that $4\bar{v}_1 + 5\bar{v}_2 - 3\bar{v}_3 = \bar{0}$. Find a basis for H (without doing EROs).

$\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LD, so cannot be a basis for H.

We can see that $\bar{v}_i \neq c\bar{v}_j$ for $i,j = 1,2,3$, $i \neq j$.

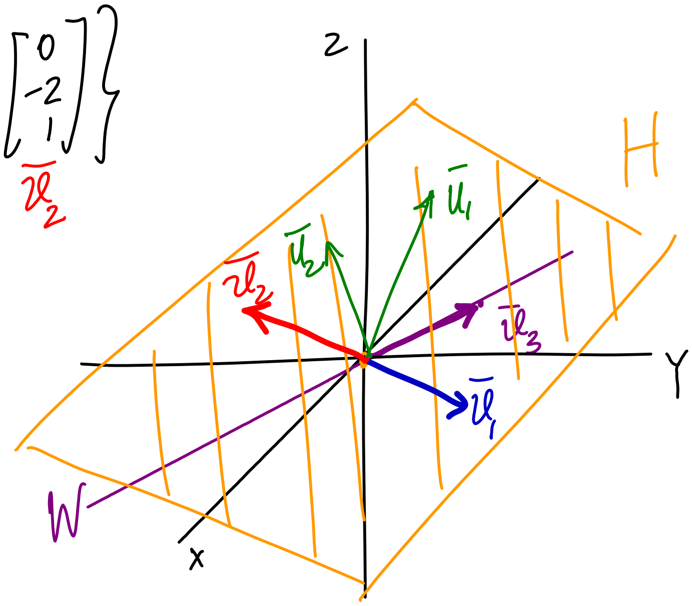
Hence $\{\bar{v}_1, \bar{v}_2\}$ is LI, and $\text{span}\{\bar{v}_1, \bar{v}_2\} = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ (since $4\bar{v}_1 + 5\bar{v}_2 - 3\bar{v}_3 = \bar{0}$). Hence $\{v_1, v_2\}$ span H, and hence is a basis for H.

Similarly, we can pick $\{\bar{v}_1, \bar{v}_3\}$ or $\{\bar{v}_2, \bar{v}_3\}$ as bases for H.

Here, $\dim H$ = dimension of $H = 2$.

$$H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$\dim H = 2$ as $\left\{ \underset{\bar{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}, \underset{\bar{v}_2}{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}} \right\}$ is a basis.



$$W = \operatorname{span}\left\{ \underset{\bar{v}_3}{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}} \right\} \qquad \dim W = 1.$$

$\operatorname{Span}\{\bar{u}_1, \bar{u}_2\} = H$

<u>Prob 12, pg 261</u>

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$$

Find the dimension of the subspace spanned by these vectors. $\to H$

Note: $\dim H < 4$, as 4 vectors in $\mathbb{R}^3$ are LD.

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 + 2R_3} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 0 & 0 & 8 \\ 0 & 1 & 5 & 7 \end{bmatrix}$$

Hence $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}$ is a basis for H. $\dim H = 3$.

If $H = \{\bar{0}\}$, $\dim H = 0$, as $\{\bar{0}\}$ is not LI.

Basis Theorem (Theorem 12, DL-LAA pg 259)

Let V be a vector space with $\dim V = p \geq 1$. Any LI subset of exactly p elements in V is a basis for V. Also, any subset of p elements that span V is a basis for V.

* $\dim V = p \rightsquigarrow$ # elements in a basis
* basis has LI elements
* basis spans V.

$\rightarrow$ two of these three results imply the third.

If $A \in \mathbb{R}^{m \times n}$

$\dim \operatorname{Col} A = \#$ pivot columns in $A \leq m$

$\dim \operatorname{Nul} A = \#$ free variables in $A \leq n$

(we can have all n variables free if A is the $m \times n$ zero matrix).

<u>Prob 21, pg 261</u>

Show that $\{1, 2t, -2+4t^2, -12t+8t^3\}$ forms a basis for $\mathbb{P}_3$.

Hermite polynomials

Note: $\dim \mathbb{P}_3 = 4$. In general, $\dim \mathbb{P}_n = n+1$.

"corresponds to"

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \Longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Hence $\{1, 2t, -2+4t^2, -12t+8t^3\} \Longleftrightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix} \right\}$.

$$\left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \ a_j \in \mathbb{R},\ j = 0,1,2,3 \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix} \right\},$$

○ : pivots

and these four vectors are LI.

Hence $\{1, 2t, -2+4t^2, -12t+8t^3\}$ span $\mathbb{P}_3$.

(prob 15, pg 181, section 2.9)

$A \in \mathbb{R}^{3\times5}$, has 3 pivot columns.

Is $\text{Col}A = \mathbb{R}^3$? Is $\text{Nul}\,A = \mathbb{R}^2$? What are

$\dim \text{Col}\,A$ and $\dim \text{Nul}\,A$?

$\text{Col}A = \mathbb{R}^3$. The columns of A are in $\mathbb{R}^3$ as

A is 3×5. Since A has 3 pivots, there is

a pivot in every row, hence the columns span

$\mathbb{R}^3$, i.e., $\text{Col}\,A = \mathbb{R}^3$.

$\text{Nul}\,A$ is a subspace of $\mathbb{R}^5$, and hence

cannot be $\mathbb{R}^2$. But $\dim \text{Nul}\,A = 2$ here.

as there are 2 free variables.

$\dim \text{Col}A = \#$ pivot columns $= 3$.

(Section 4.6)

Def $A \in \mathbb{R}^{m\times n}$, $\dim \text{Col}\,A$ is called the

rank of the matrix A (written as $\text{rank}\,A$ or $\text{rank}(A)$).

So, $\text{rank}\,A = \#$ pivot columns in A.

Rank Theorem (Theorem 14, DL-LAA pg 265)

If $A \in \mathbb{R}^{m\times n}$, then

$$\boxed{\text{rank } A + \dim \text{Nul } A = n}$$

Prob Create a 3×4 matrix A of rank 2. What is dim Nul A?

$$A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ has 2 pivots and hence rank $A = 2$

Since rank A + dim Nul $A = n = 4$,

dim Nul $A = 2$.

The columns 1, 3, 5, and 6 of A are LI, and rank $A = 4$. Explain why these four columns must be a basis for Col A.

Since rank A = dim Col $A = 4$, any 4 LI columns of A form a basis for Col A (Basis theorem).

Invertible Matrix Theorem (IMT)

See Lecture 17 for original statement of IMT.

(a) $A \in \mathbb{R}^{n\times n}$ is invertible

(m) Columns of A form a basis for $\mathbb{R}^n$.

(n) $\text{Col}\, A = \mathbb{R}^n$

(o) $\dim \text{Col}\, A = n$

(p) $\text{rank}\, A = n$

(q) $\text{Nul}\, A = \{\bar{0}\}$

(r) $\dim \text{Nul}\, A = 0$.

MATH230 - Lecture 27 (04/19/2011)

Eigenvalues and Eigenvectors (Chapter 5)

$A \in \mathbb{R}^{n\times n}$ LT $\bar{x} \longmapsto A\bar{x}$ We know how to find a basis for the range of this LT $(= \text{Col } A)$.

What can we say about vectors $\bar{x}$ for which $T(\bar{x})$ is just a scalar multiple of $\bar{x}$?

$$A\bar{x} = \lambda \bar{x} \text{ for some } \lambda \in \mathbb{R}.$$

Def An n-vector $\bar{x}$ is an **eigenvector** of A if it is non-zero, and for some scalar λ, we have $A\bar{x} = \lambda\bar{x}$. λ is called an **eigenvalue** of A, and $\bar{x}$ is an eigenvector corresponding to the eigenvalue λ.

Note: λ can be zero. $\bar{x}$ has to be nonzero.

$\bar{x} = \bar{0}$ always satisfies $A\bar{x} = \lambda\bar{x}$. Hence we only look for non-zero $\bar{x}$ (for eigenvectors).

non-trivial solutions to some homogeneous system.

Some 2x2 examples

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ The matrix of the LT $T: \mathbb{R}^2 \to \mathbb{R}^2$ which reflects on the $45°$ line $y = x$.

Find $\bar{x} \neq \bar{0}$ and λ such that $A\bar{x} = \lambda\bar{x}$. There are two unknowns: $\bar{x}$ and λ.

$A\bar{x} - \lambda\bar{x} = \bar{0}$, i.e., $A\bar{x} - \lambda I\bar{x}$ where I is the 2×2 identity matrix.

$\Rightarrow (A - \lambda I)\bar{x} = \bar{0}$ To get $\bar{x} \neq \bar{0}$ as solutions, we need $A - \lambda I$ not invertible, i.e., $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}. \Rightarrow \det(A - \lambda I) = \lambda^2 - 1$$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$ There are two eigenvalues, $\lambda = +1$, $\lambda = -1$.

For $\lambda = 1$, we can find an eigenvector by solving $(A - \lambda I)\bar{x} = \bar{0}$ i.e, $(A - I)\bar{x} = \bar{0} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

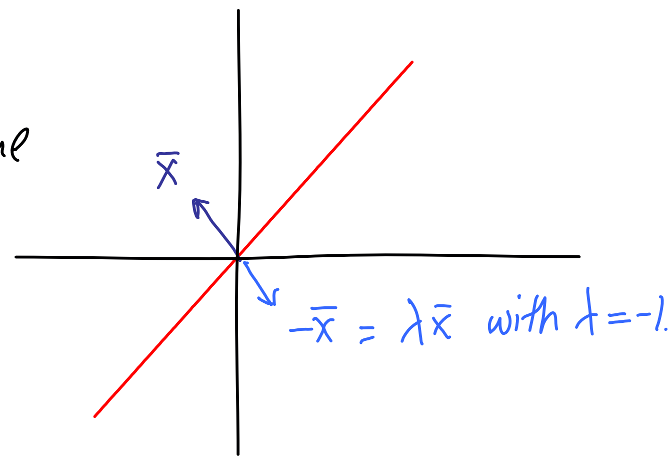
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow[R_1 \rightleftarrows R_2]{R_1 + R_2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$ x_2 free, $x_1 = x_2$ $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$, $x_2 \in \mathbb{R}$

So, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

$\lambda = -1$: $\quad (A - \lambda I)\bar{x} = \bar{0} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad x_2$ free. $\quad x_1 = -x_2 \quad \bar{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2, \; x_2 \in \mathbb{R}$.

So, $\bar{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -1$.



(b) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$

$(3-\lambda)^2 - 1 = 0$

$3+3$

$\lambda^2 - 6\lambda + \underbrace{(9-1)}_{\det A = 3\times 3 - 1\times 1} = 0$

$(\lambda - 2)(\lambda - 4) = 0$

$\lambda = 2, 4$ are eigenvalues.

In general, for $A \in \mathbb{R}^{2\times 2}$, the equation for λ looks like $\lambda^2 - \underbrace{(\text{sum of entries in diagonal of } A)}_{\text{trace}(A)}\lambda + \det A = 0$

trace(A) → sum of diagonal entries of an $n\times n$ matrix A.

(c) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation about origin by 90° ccw.

$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0 \qquad \lambda^2 + 1 = 0$

No real eigenvalues

Result: If A is symmetric, i.e., $A^T = A$, or $A_{ij} = A_{ji}$ for all i, j, then A has only real eigenvalues.

If A is antisymmetric, i.e., $A^T = -A$, then all eigenvalues of A are imaginary.

Note: $A_{ij} = -A_{ji}$ if A is antisymmetric. Hence when $i = j$, $A_{ii} = -A_{ii} \implies A_{ii} = 0$. Hence antisymmetric matrices have zeroes on the diagonal.

We will worry only about symmetric matrices, and hence with real eigenvalues, in Math 230.

Prob 6 Page 308

Is $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ an eigen vector of $A=\begin{bmatrix}3&6&7\\3&3&7\\5&6&5\end{bmatrix}$? If yes, find the corresponding eigenvalue.

$$\bar{u}=\begin{bmatrix}1\\-2\\1\end{bmatrix}. \qquad A\bar{u}=\begin{bmatrix}3&6&7\\3&3&7\\5&6&5\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}-2\\4\\-2\end{bmatrix}=-2\begin{bmatrix}1\\-2\\1\end{bmatrix}=-2\bar{u}.$$

So, $\bar{u}$ is an eigenvector of A with $\lambda=-2$ being the eigenvalue.

Prob 8 pg 308 Is $\lambda=3$ an eigenvalue of $A=\begin{bmatrix}1&2&2\\3&-2&1\\0&1&1\end{bmatrix}$? If yes, find a corresponding eigenvector.

Check if $\det(A-\lambda I)$ is zero.

$$A-3I=\begin{bmatrix}-2&2&2\\3&-5&1\\0&1&-2\end{bmatrix} \qquad \det(A-\lambda I)=(-1)^{3+2}\cdot 1\cdot\begin{vmatrix}-2&2\\3&1\end{vmatrix}$$

$$+(-1)^{3+3}\cdot(-2)\cdot\begin{vmatrix}-2&2\\3&-5\end{vmatrix}$$

$$=8-8=0. \text{ Yes!}$$

To find an eigenvector, find a non-trivial solution to $(A-\lambda I)\bar{x}=\bar{0}$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 \times -\frac{1}{2}} \begin{bmatrix} 1 & -1 & -1 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 + 2R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_2 \rightleftarrows R_3]{R_1 + R_3} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$ x_3 free, $\bar{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} x_3$, $x_3 \in \mathbb{R}$.

So, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 3$.

<u>Def</u>: **Eigenspace** of A corresponding to eigenvalue λ is the set of all eigenvectors corresponding to the eigenvalue λ plus the **zero vector** (eventhough zero vector is not an eigen vector).

<u>Prob 10, pg 308</u> Find a basis for the eigenspace of $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 4$.

Find a basis for $Nul(A - \lambda I)$.

$$A - \lambda I = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} \xrightarrow{R_2 - \frac{2}{3}R_1} \begin{bmatrix} 6 & -9 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \times \frac{1}{6}} \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$$

$\bar{x} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} x_2$, $x_2 \in \mathbb{R}$. Hence $\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$ is a basis for the corresponding eigenspace.

dimension of eigenspace is 1.

Eigenvalues of triangular matrices

Recall: If A is a triangular matrix, $\det A$ is just the product of entries on the diagonal.

$$A = \begin{bmatrix} a & * & * & \cdots & * \\ 0 & b & & & \vdots \\ 0 & & c & & \vdots \\ \vdots & & & \ddots & * \\ 0 & 0 & 0 & & m \end{bmatrix}$$ (upper triangular)

$$\det A = a \cdot b \cdot c \cdots m$$

What about $\det(A-\lambda I)$?

$$A-\lambda I = \begin{bmatrix} a-\lambda & & * \\ & b-\lambda & \\ 0 & & \ddots & \\ & & & m-\lambda \end{bmatrix},$$ hence

$$\det(A-\lambda I) = (a-\lambda)(b-\lambda)\cdots(m-\lambda).$$

Hence $a, b, c, \ldots, m$ are roots of the equation

$\det(A-\lambda I)=0$. So,

Eigenvalues of triangular matrices are the entries on the diagonal.

Def: $\det(A-\lambda I)$ is called the **characteristic polynomial** of A, and $\det(A-\lambda I)=0$ is the **characteristic equation** of A. (In MATLAB, look for charpoly(·)). eig(·) gives both eigenvalues and eigenvectors).

Prob 10 pg 317 Find the characteristic polynomial of

$$A=\begin{bmatrix}0&3&1\\3&0&2\\1&2&0\end{bmatrix}.$$

Note: A is symmetric here.

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda&3&1\\3&-\lambda&2\\1&2&-\lambda\end{vmatrix}=(-\lambda)\begin{vmatrix}-\lambda&2\\2&-\lambda\end{vmatrix}-3\begin{vmatrix}3&2\\1&-\lambda\end{vmatrix}$$

$$+1\begin{vmatrix}3&-\lambda\\1&2\end{vmatrix}$$

$$=(-\lambda)(\lambda^2-4)-3(-3\lambda-2)+(6+\lambda)$$

$$\vdots$$

MATH230 — Lecture 28 (04/21/2011)

Characteristic polynomial of $A \in \mathbb{R}^{n\times n}$ is $\det(A-\lambda I)$

Characteristic equation : $\det(A-\lambda I)=0$.

Prob 10 pg 317 Find the characteristic polynomial of

Note: A is symmetric here.

$$A=\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = (-1)^{2+1}\cdot 3\cdot \begin{vmatrix} 3 & 1 \\ 2 & -\lambda \end{vmatrix} + (-1)^{2+2}(-\lambda)\cdot\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$+ (-1)^{2+3}(2)\cdot\begin{vmatrix} -\lambda & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -3(-3\lambda-2) - \lambda(\lambda^2-1) - 2(-2\lambda-3)$$

$$= -\lambda^3 + (9+1+4)\lambda + (6+6) = -\lambda^3 + 14\lambda + 12$$

Eigenspace corresponding to eigenvalue λ

def 1: Vector space spanned by all linearly independent eigenvectors corresponding to eigenvalue λ.

def 2: Collection of all eigenvectors plus the zero vector.

even though $\bar{0}$ is not an eigenvector

def 3: Nullspace of $(A-\lambda I)$.

Prob 20 pg 308

$A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Without calculation, find an eigenvalue and two linearly independent eigenvectors (corresponding to that eigenvalue) of A.

Notice that $A\bar{x} = \bar{0}$ will have non-trivial solutions. We can have only one pivot, and hence get two free variables.

Hence $\lambda = 0$ is an eigenvalue, because with $\lambda = 0$, $A\bar{x} = \lambda\bar{x}$ is the same as $A\bar{x} = \bar{0}$. Hence all non-trivial solutions to $A\bar{x} = \bar{0}$ are eigenvectors of A corresponding to $\lambda = 0$.

We can choose $\bar{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\bar{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ as two LI eigenvecters.

Another eigenvalue is $\lambda' = 15$, as $\bar{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ gives

$A\bar{x}_3 = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15\bar{x}_3$. So, $\bar{x}_3$ is an eigenvector corresponding to eigenvalue $\lambda = 15$.

Prob 16 Pg 318

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$$

List all eigenvalues along with their multiplicities.

Since A is triangular, the eigenvalues are on the diagonal. They are $5(1)$, $-4(1)$, and $1(2)$, with the multiplicities listed in $(\cdot)$.

Def: The **multiplicity** of an eigenvalue is the # times it appears as a root of the characteristic equation.

e.g., $(\lambda-3)^2(\lambda+2)(\lambda+4)^3 = 0 \longrightarrow$ characteristic equation

The eigenvalues and their multiplicities are

$3(2)$, $-2(1)$, and $-4(3)$.

Prob 26 Page 309

Show: If A^2 is the zero matrix, then the only eigenvalue of A is 0.

λ is an eigenvalue of $A \Longrightarrow A(\underbrace{A\bar{x} = \lambda\bar{x}})$ $\quad \bar{x} \neq \bar{0}$, as it's an eigenvector

$$\Rightarrow \underbrace{A^2}_{\bar{0}}\bar{x} = \lambda\underbrace{A\bar{x}}_{\lambda\bar{x}} \quad \Rightarrow \bar{0} = \lambda\lambda\bar{x} = \lambda^2\bar{x}$$

$\Rightarrow \lambda^2\bar{x} = \bar{0}$ with $\bar{x} \neq \bar{0}$. Hence we must have $\lambda^2 = 0$, i.e., $\lambda = 0$.

Similarity

Def $A, B \in \mathbb{R}^{n\times n}$. A is similar to B if there is an invertible matrix P such that

$$B = P^{-1}AP \quad \text{and} \quad A = PBP^{-1}.$$

Result If A is similar to B, then A and B have the same characteristic polynomial and hence the same set of eigenvalues.

Proof $B = P^{-1}AP$ for invertible P.

Want to show $\det(A-\lambda I) = \det(B-\lambda I)$

$$B-\lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda \underbrace{P^{-1}P}_{I}$$

$$= P^{-1}(AP-\lambda P) = P^{-1}(A-\lambda I)P$$

So, $B-\lambda I$ is similar to $A-\lambda I$.

Taking determinants, we get

$$\det(B-\lambda I) = \det\left(P^{-1}(A-\lambda I)P\right)$$

$$= \det(P^{-1})\det(A-\lambda I)\cdot\det(P)$$ as $\det(AB) = \det A\cdot\det B$

$$= \frac{1}{\cancel{\det(P)}}\cdot\det(A-\lambda I)\cdot\cancel{\det(P)}$$ as P is invertible $\det(P)\neq 0$

$$\Rightarrow \det(B-\lambda I) = \det(A-\lambda I).$$

The QR algorithm to estimate the eigenvalues of A

Prob 23 pg 318 If $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.

Note that both Q and R are $n\times n$ when $A\in\mathbb{R}^{n\times n}$

Want to show: There is invertible matrix P such that

$$A_1 = P^{-1}AP$$

$$A_1 = RQ = \underbrace{Q^{-1}\overbrace{Q}}_{I\text{, as } Q \text{ is invertible}}RQ = Q^{-1}(QR)Q = Q^{-1}AQ$$

So A_1 and A are similar.

The idea of QR algorithm: Factorize A into $Q_1 R_1$ where $Q_1^T = Q_1^{-1}$ and R_1 is upper triangular. So, $A = Q_1 R_1$. Write $A_1 = R_1 Q_1$, and repeat, i.e., write $A_1 = Q_2 R_2$ with $Q_2^T = Q_2^{-1}$, R_2 upper triangular. Then set $A_2 = R_2 Q_2$, and so on.

After sufficient # iterations (k), the diagonal entries of R_k will be close to the eigenvalues of A. Higher the k, closer the diagonal entries of R_k are to eigenvalues of A. Hence the use of the word "estimate".

MATH230 — Lecture 29 (04/26/2011)

QR algorithm $(A \in \mathbb{R}^{n\times n})$

$A = Q_1R_1$ where $Q_1^T = Q_1^{-1}$ and R_1 is upper triangalur.

set $A_1 = R_1Q_1$

Decompose $A_1 \rightarrow Q_2R_2$, write $A_2 = R_2Q_2$, and so on.

Diagonal entries of A_k approximate eigenvalues of A.

In MATLAB, use the function qr to find the Q and R.

(MATLAB demo $\longrightarrow$ see course web page)

Prob 18 Pg 318

It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in matrix A below for which the eigenspace corresponding to $\lambda=5$ is 2-dimensional.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $\lambda = 5$, $A - \lambda I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 0 & 0 & 6-h & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \rightleftarrows R_1} \begin{bmatrix} 0 & -2 & h & 0 \\ 0 & 0 & 6-h & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We need $h = 6$ so that there are two free variables, and hence the dimension of the eigenspace is 2.

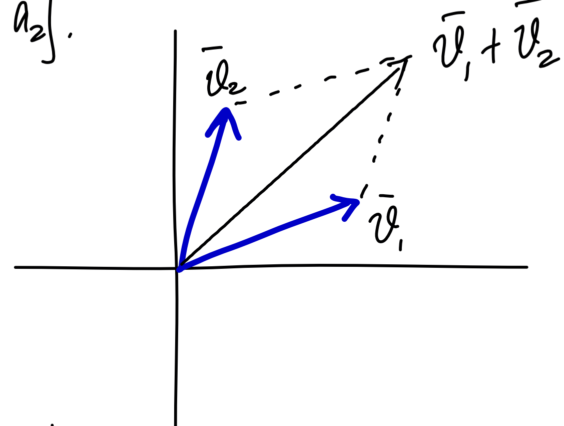
(Notice that since A is 4×4, we need two pivots so that $\dim \operatorname{Nul} A = n - 2 = 4 - 2$ is $= 2$).

We could have made the same conclusion by looking at the columns of A. We need two pivots. The fourth column has one pivot. Between columns 2 & 3, we need one pivot, which is possible when $h = 6$.

Prob 22, pg 318 (True/False)

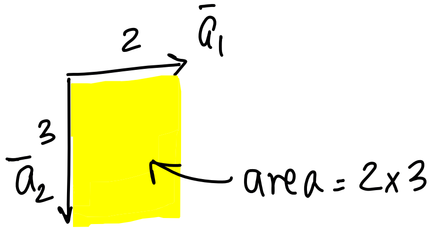
(a) If $A \in \mathbb{R}^{3\times 3}$, then $\det A$ is the volume of the parallelepiped determined by $\bar{a}_1, \bar{a}_2, \bar{a}_3$ where $A = [\bar{a}_1\ \bar{a}_2\ \bar{a}_3]$.

In 2D, two LI vectors determine a parallelogram.



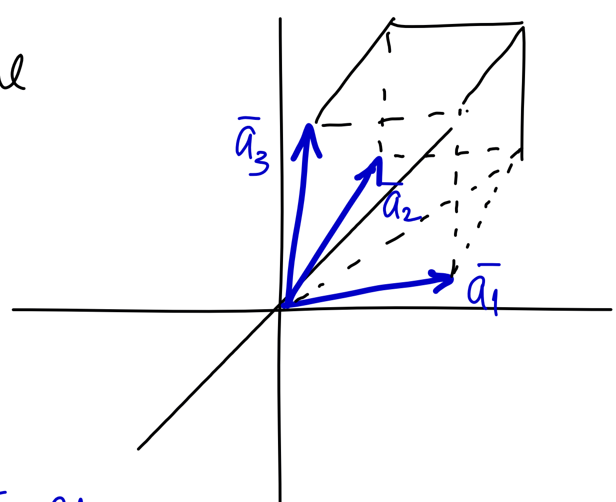
Another example: $\bar{a}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, $A = [\bar{a}_1\ \bar{a}_2]$.

$\det A = \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -6$



So $|\det A|$ gives the area of the rectangle.

In general, $|\det A|$ gives the area of the parallelogram formed by $\bar{a}_1$ and $\bar{a}_2$. In fact, in arbitrary dimensions, $|\det A|$ gives the volume of the parallelepiped formed by $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$.



As given, the statement is FALSE, as $|\det A|$, and not $\det A$, gives the volume.

When $A=\begin{bmatrix}1&2\\3&6\end{bmatrix}$, $\det A=0$. Indeed, the parallelogram generated by $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\6\end{bmatrix}$ is just a squished line segment. Hence its area is zero.

(d) A row replacement ERO on A does not change its eigenvalues.

FALSE. Let $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$. $\det(A-\lambda I) = (1-\lambda)(1-\lambda) = \lambda^2-2\lambda+1$

$$A \xrightarrow{R_1+R_2} \begin{bmatrix}2&1\\1&1\end{bmatrix} = A' \quad \det(A'-\lambda I) = (2-\lambda)(1-\lambda) - 1\times 1$$
$$= \lambda^2-3\lambda+2-1 = \lambda^2-3\lambda+1$$

So, eigenvalues of A and A' are not same.

But we could have examples where replacement EROs do not change the eigenvalues.

e.g., $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\det(A-\lambda I) = (1-\lambda)^2$

$$A \xrightarrow{R_1+R_2} \begin{bmatrix}1&1\\0&1\end{bmatrix} = A'. \quad \det(A'-\lambda I) = (1-\lambda)^2$$

Theorem 2 (Section 5.1)

If $\bar{v}_1, \dots, \bar{v}_r$ are eigenvectors of a matrix A corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then $\{\bar{v}_1, \dots, \bar{v}_r\}$ is LI.

Proof for 2 vectors: Let $\bar{v}_1, \bar{v}_2$ be eigenvectors corresponding to λ_1, λ_2 with $\lambda_1 \neq \lambda_2$. Then we need to show $\{\bar{v}_1, \bar{v}_2\}$ is LI, i.e., $\bar{v}_1 \neq c\,\bar{v}_2$ for $c \in \mathbb{R}$.

Assume $\{\bar{v}_1, \bar{v}_2\}$ is LD, i.e., $\bar{v}_1 = c\,\bar{v}_2$.

$$A\left(\bar{v}_1 = c\,\bar{v}_2\right)$$

$\Rightarrow A\bar{v}_1 = c\,A\bar{v}_2$ But $A\bar{v}_1 = \lambda_1 \bar{v}_1$ and $A\bar{v}_2 = \lambda_2 \bar{v}_2$.

$\Rightarrow \lambda_1 \bar{v}_1 = c\,\lambda_2 \bar{v}_2$

Case 1: $\lambda_1 = 0$ or $\lambda_2 = 0$; Say $\lambda_1 = 0$. Then we get

$0 = c\lambda_2 \bar{v}_2$ i.e., $\bar{v}_2 = \bar{0}$.

MATH230 — Lecture 30 (04/28/2011)

Eigenvectors corresponding to distinct eigenvalues are LI.

$\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p, \bar{v}_{p+1}, \dots$ are eigenvectors of A corresponding to distinct eigenvalues $\underbrace{\lambda_1, \lambda_2, \dots, \lambda_p, \lambda_{p+1}, \dots}_{\lambda_i \neq \lambda_j \ \forall i,j}$.

Case of $\bar{v}_1, \bar{v}_2$

Assume $\{\bar{v}_1, \bar{v}_2\}$ is LD. Then $\bar{v}_1 = c\bar{v}_2$ for $c \in \mathbb{R}$.

$c \neq 0$ as both $\bar{v}_1$ and $\bar{v}_2$ are non-zero vectors

$A(\bar{v}_1 = c\bar{v}_2)$

$\Rightarrow A\bar{v}_1 = cA\bar{v}_2$ but $A\bar{v}_1 = \lambda_1\bar{v}_1$, $A\bar{v}_2 = \lambda_2\bar{v}_2$

$\Rightarrow \lambda_1\bar{v}_1 = c\lambda_2\bar{v}_2$

Case 1 One of λ_1, λ_2 is zero, say $\lambda_1 = 0$.

We get $\bar{0} = c\lambda_2\bar{v}_2 \Rightarrow \bar{v}_2 = \bar{0}$, but $\bar{v}_2 \neq \bar{0}$ as it is an eigenvector. We get a contradiction.

Case 2 $\lambda_1 \neq 0$, $\lambda_2 \neq 0$

$\Rightarrow \bar{v}_1 = \left(c\frac{\lambda_2}{\lambda_1}\right)\bar{v}_2 = c'\bar{v}_2$ where $c' = c\frac{\lambda_2}{\lambda_1} \neq c$ as $\lambda_1 \neq \lambda_2$

But $\bar{v}_1 = c\bar{v}_2$, hence we must have $\bar{v}_1 = \bar{v}_2 = \bar{0}$, giving again a contradiction.

Note: We must have $\lambda_1 \neq \lambda_2$ as they are distinct.

Proof in general

Let $\{\bar{v}_1, \dots, \bar{v}_p\}$ be LI, but $\bar{v}_{p+1}$ be a linear combination of $\bar{v}_1, \dots, \bar{v}_p$ for some $p \geq 2$.

$$\Rightarrow A\left(\bar{v}_{p+1} = c_1 \bar{v}_1 + \dots + c_p \bar{v}_p\right) \quad \text{———} \quad (1)$$

$$\Rightarrow A\bar{v}_{p+1} = c_1 A\bar{v}_1 + \dots + c_p A\bar{v}_p$$

$$\Rightarrow \lambda_{p+1} \bar{v}_{p+1} = c_1 \lambda_1 \bar{v}_1 + \dots + c_p \lambda_p \bar{v}_p \quad \text{———} \quad (2)$$

as $A\bar{v}_j = \lambda_j \bar{v}_j \ \forall j$

$(2) - \lambda_{p+1}(1)$ gives

$$\bar{0} = c_1 \underbrace{(\lambda_1 - \lambda_{p+1})}_{\neq 0} \bar{v}_1 + \dots + c_p \underbrace{(\lambda_p - \lambda_{p+1})}_{\neq 0} \bar{v}_p$$

as λ_j's are distinct

Since $\{\bar{v}_1, \dots, \bar{v}_p\}$ is LI, we must have $c_1 = \dots = c_p = 0$ here. But then $\bar{v}_{p+1} = c_1 \bar{v}_1 + \dots + c_p \bar{v}_p = \bar{0}$, which cannot be true, as $\bar{v}_{p+1}$ is an eigen vector. So we get a contradiction.

Review for Final (Problems from Practice Final).

Prob 3

Rank A = # pivot columns. We need a 3x3 matrix with 2 pivots here.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ has rank 2, as two columns are LI, but column 3 is the same as column 2.

take first column of A

$\bar{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \text{Nul } A$, as $A\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \bar{0}$.

Another choice is $\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, as $A\bar{b} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \neq \bar{0}$.

add all columns of A

Prob 8

$$\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}}_{\bar{v}_1} a + \underbrace{\begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}}_{\bar{v}_2} b + \underbrace{\begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}}_{\bar{v}_3} c$$

So, the set W of all vectors of the given form can be written as

$W = \text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$.

Notice that $\bar{v}_1 - 2\bar{v}_2 = \bar{v}_3$.

If you do not notice this relation, just check by the usual method if $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LI or not.

$$\begin{bmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ -1 & -4 & 7 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3+R_1 \\ R_4-3R_1}]{R_2-2R_1} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \\ 0 & -6 & 12 \\ 0 & 7 & -14 \end{bmatrix} \xrightarrow[R_4-(7/9)R_2]{R_3+\frac{6}{9}R_2} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But $\bar{v}_i \neq c\bar{v}_j$ for $i,j = 1,2,3$, $i \neq j$ here. Hence $\{\bar{v}_1, \bar{v}_2\}$, $\{\bar{v}_1, \bar{v}_3\}$, $\{\bar{v}_2, \bar{v}_3\}$ are all bases for W.

Prob 2

A has to be 2×2.

Given: $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $A\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\Rightarrow A\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \qquad \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\right) = 1 \neq 0$$

$$\text{Hence } \left(\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\right)^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}.$$

$$\text{Hence } A\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}}_{I} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}.$$

Prob 6 $p_1(t) = 1+t$, $p_2(t) = 1-t$, $p_3(t) = -4$, $p_4(t) = t+t^2$, $p_5(t) = 1+2t-2t^2$.

$\mathbb{P}_n$ is the set of all polynomials with degree upto n.

$$p(t) = a_0 + a_1 t + \dots + a_n t^n \equiv \bar{v} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

go to a vector corresponding to each polynomial

Here, we get

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_5 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}.$$

$$A = [\bar{v}_3 \; \bar{v}_1 \; \bar{v}_2 \; \bar{v}_4 \; \bar{v}_5] = \begin{array}{c} \begin{bmatrix} -4 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} p_3 & p_1 & p_2 & p_4 & p_5 \end{array} \end{array}$$

Hence $\{p_1(t), p_3(t), p_4(t)\}$ is a basis for H. Similarly, $\{p_2(t), p_3(t), p_4(t)\}$ is also a basis.

Note: We could take any three LI vectors out of the five, and pick the corresponding polynomials for a basis here.

We could have avoided the 4^{th} entry — which is uniformly zero here — and reached the same conclusions.

Prob 12, True/False

(a) FALSE. $A=\begin{bmatrix}1&2\\1&2\end{bmatrix}$, $B=\begin{bmatrix}1&2\\0&0\end{bmatrix} \Rightarrow \text{rref}(A)=\text{rref}(B)=B$.

But Col A is $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, while Col B $=\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

(b) TRUE. All columns of A are pivot columns. Hence there are no free variables. So $A\bar{x}=\bar{0}$ has only the trivial solution.

(c) TRUE. $(AB)^{-1}=B^{-1}A^{-1}$ Product of two matrices is invertible iff both are invertible.

Since AB^{-1} is invertible, so are A and B^{-1}.

Hence both A and B are invertible. So $(AB)^{-1}=B^{-1}A^{-1}$.

(d) FALSE. The result holds only if the plane passes through the origin.

(e) TRUE. $A^{-1}(A\bar{x}=\lambda\bar{x})$ A^{-1} exists

$\bar{x}=\lambda A^{-1}\bar{x}$. Since $\lambda\neq 0$, we get $A^{-1}\bar{x}=\left(\frac{1}{\lambda}\right)\bar{x}$

So $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. $\bar{x}$ is an eigenvector of A and A^{-1} here.