MATH 230 – Lecture 10 (02/10/2011)

A note on writing general solutions in parametric vector form.

e.g., $\bar{x} \in \mathbb{R}^4$ with x_3, x_4 free:

$$\bar{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} x_4, \quad x_3, x_4 \in \mathbb{R} \qquad \text{OR}$$

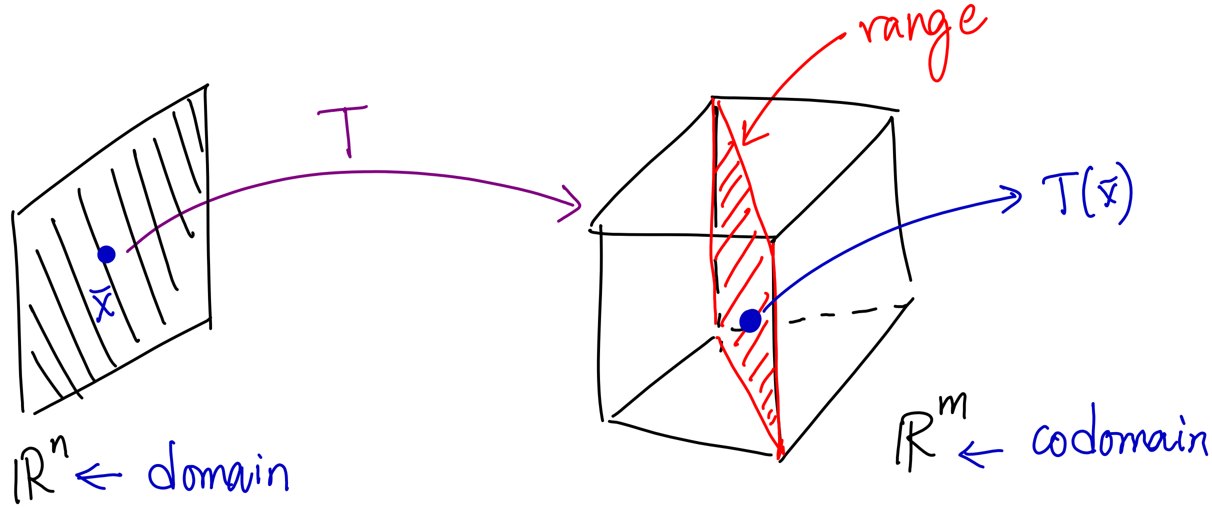
$$\bar{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} t, \quad s, t \in \mathbb{R}$$

need this specification

Back to Linear transformations

Def: A <u>transformation</u> T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns every vector $\bar{x}$ in $\mathbb{R}^n$ a vector $T(\bar{x})$ in $\mathbb{R}^m$.

$T: \mathbb{R}^n \to \mathbb{R}^m$



$T(\bar{x})$ is the image of $\bar{x}$ under T.

$$T: \underset{\text{domain}}{\mathbb{R}^n} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^m}$$

Range is the set of all images (of the form $T(\bar{x})$) under T. It is a subset of the codomain; it could be the entire codomain in some cases.

Linear Transformations

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation (LT) if

(i) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v}$ in $\mathbb{R}^n$; and

(ii) $T(c\bar{u}) = c\,T(\bar{u})$ for all $\bar{u} \in \mathbb{R}^n$, and scalar c.

In words, T is linear if it preserves vector addition and scalar-vector multiplication.

Equivalently, we can say that T is linear if the image of a sum is the sum of images, and if image of scalar $\times$ vector is scalar $\times$ image of vector, under T.

As a consequence of Rules (i) and (ii), we get

$T(\bar{0}) = \bar{0}$ and

$T(c\bar{u}+d\bar{v}) = cT(\bar{u})+dT(\bar{v})$ for $c, d \in \mathbb{R}$ and $\bar{u}, \bar{v} \in \mathbb{R}^n$.

<u>Prob 17, pg 80</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ is an LT.

$\bar{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $T(\bar{u}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the images under T of $3\bar{u}$, $2\bar{v}$, and $3\bar{u}+2\bar{v}$.

$$T(3\bar{u}) = 3T(\bar{u}) = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$T(2\bar{v}) = 2T(\bar{v}) = 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$T(3\bar{u}+2\bar{v}) = 3T(\bar{u})+2T(\bar{v}) = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\times 2+2\times -1 \\ 3\times 1+2\times 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

Equivalently, since we just found $T(3\bar{u})$ and $T(2\bar{v})$, we can write

$$T(3\bar{u}+2\bar{v}) = T(3\bar{u})+T(2\bar{v}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

Prob 24, pg 81

Suppose $\bar{v}_1, \dots, \bar{v}_p$ span $\mathbb{R}^n$, and $T: \mathbb{R}^n \to \mathbb{R}^n$ be an LT. It is known that $T(\bar{v}_i) = \bar{0}$ for all $i = 1, \dots, p$. Show that T is the zero-transformation.

$T(\bar{x}) = \bar{0}$ for all $\bar{x} \in \mathbb{R}^n$.

We want to show that $T(\bar{x}) = \bar{0}$ for all $\bar{x} \in \mathbb{R}^n$.

$\bar{v}_1, \dots, \bar{v}_p$ span $\mathbb{R}^n$. Hence any $\bar{x} \in \mathbb{R}^n$ can be written as $\bar{x} = c_1 \bar{v}_1 + \dots + c_p \bar{v}_p$ for scalars $c_1, \dots, c_p$.

Hence
$$
\begin{aligned}
T(\bar{x}) &= T(c_1 \bar{v}_1 + \dots + c_p \bar{v}_p) \\
&\cancel{= T(c_1 \bar{v}_1) + \dots + T(c_p \bar{v}_p)} \quad \text{could skip.} \\
&= c_1 T(\bar{v}_1) + \dots + c_p T(\bar{v}_p) \quad \text{as } T \text{ is linear} \\
&= c_1 \bar{0} + \dots + c_p \cdot \bar{0} \quad \text{as } T(\bar{v}_i) = \bar{0} \text{ for all } i. \\
&= \bar{0}
\end{aligned}
$$

Prob 32, pg 81

$$T(x_1, x_2) = (4x_1 - 3x_2,\ 3|x_2|)$$

absolute value of x_2

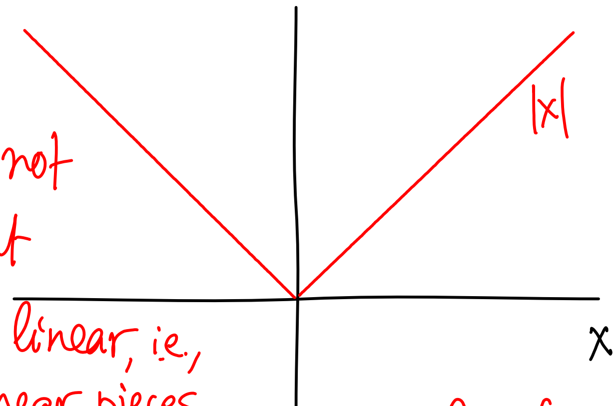
Show that T is not a linear transformation.

Show that T violates one of the properties of LTs.

Consider

$$\bar{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\quad \bar{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$|x|$ is not linear, but is piecewise linear, ie, made of linear pieces.



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$T(\bar{u}) = (4\times 2 - 3\times 1,\ 3|1|) = (9, 3)$$

$$T(\bar{v}) = (4\times 2 - 3\times -1,\ 3|-1|) = (11, 3)$$

$$\bar{u} + \bar{v} = \begin{bmatrix} 2+2 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

$$T(\bar{u}+\bar{v}) = (4\times 4 - 3\times 0,\ 3|0|) = (16, 0)$$

$$T(\bar{u}) + T(\bar{v}) = \begin{bmatrix} 9+11 \\ 3+3 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \end{bmatrix} \neq T(\bar{u}+\bar{v})$$

Since $T(\bar{u}+\bar{v}) \neq T(\bar{u}) + T(\bar{v})$, T is not an LT.

We could also use a simpler pair of vectors, e.g., $\bar{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, to show $T(\bar{u}) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. $T(\bar{u}) + T(\bar{v}) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, but $T(\bar{u}+\bar{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Matrix of an LT

Result: Every LT T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation $T(\bar{x}) = A\bar{x}$. (Also denoted as $\bar{x} \longmapsto A\bar{x}$).

We can find the A for LT T if we know what T does to the columns of the identity matrix.

$$\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots\ 0 & & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1 \end{bmatrix}$$

1's along the diagonal and zeros everywhere else.