MATH230 - Lecture 11 (02/15/2011)

Midterm on Tue, Mar 1 (in class).

Topics covered TBA, but should include everything covered up to Lec 12 (Thu, Feb 17).

No MATLAB-related questions!

Practice mid-term will be posted by early next week. Go through homework problems!

Review for midterm on Thursday, Feb 24.

Linear Transformations (LTs)

Prob 12, pg 80

$$A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}.$$

Is $\bar{b}$ in the range of the LT $\bar{x} \mapsto A\bar{x}$?

Reword: Does $A\bar{x} = \bar{b}$ have at least one solution?

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{array}\right] \xrightarrow[R_4+2R_1]{R_2-R_1} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{array}\right] \xrightarrow[R_4-9R_3]{R_2+3R_3} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & -18 & 11 \end{array}\right]$$

$$\xrightarrow{R_4+6R_2} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 17 \end{array}\right]$$

system is inconsistent. So $\bar{b}$ is not in the range of $\bar{x} \mapsto A\bar{x}$.

Matrix of an LT

"if and only if"

$T: \mathbb{R}^n \to \mathbb{R}^m$ is an LT $\Longleftrightarrow$ $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{m \times n}$.

Theorem 10 DL-LAA (pg 83) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be an LT. Then there exists a unique $m \times n$ matrix A such that $T(\bar{x}) = A\bar{x}$ for every $\bar{x} \in \mathbb{R}^n$. This matrix A is given as

$$A = [T(\bar{e}_1) \;\; T(\bar{e}_2) \; \cdots \; T(\bar{e}_n)], \text{ where}$$

$\bar{e}_j$ is the j^{th} unit vector in $\mathbb{R}^n$, i.e., $\bar{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← j^{th} position

Proof Let $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. We can write

$$\bar{x} = x_1 \bar{e}_1 + x_2 \bar{e}_2 + \cdots + x_n \bar{e}_n,$$

where $\bar{e}_j$ is the j^{th} unit vector.

$$\left(= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

So, $T(\bar{x}) = T(x_1\bar{e}_1 + \cdots + x_n\bar{e}_n)$

$= x_1 T(\bar{e}_1) + \cdots + x_n T(\bar{e}_n)$, as T is an LT.

$$= \left[T(\bar{e}_1) \cdots T(\bar{e}_n) \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$= A\bar{x}$ for $A = \left[T(\bar{e}_1) \cdots T(\bar{e}_n) \right]$.

We still need to show that A is unique.

Assume $T(\bar{x}) = B\bar{x}$ for some $m \times n$ matrix B.

So, $T(\bar{e}_j) = B\bar{e}_j = [\bar{b}_1\ \bar{b}_2 \cdots \bar{b}_n] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \cdot \bar{b}_1 + \cdots + 0 \cdot \bar{b}_{j-1} + 1 \cdot \bar{b}_j + 0 \cdot \bar{b}_{j+1} + \cdots + 0 \cdot \bar{b}_n$

j^{th} position

$= \bar{b}_j$

Thus $B = [\bar{b}_1 \cdots \bar{b}_n] = [T(\bar{e}_1) \cdots T(\bar{e}_n)] = A$.

So, A is unique.

e.g., Find the matrix of the LT from $\mathbb{R}^n \to \mathbb{R}^n$ given as $T(\bar{x}) = 3\bar{x}$. → scale the input vector 3 times.

$A = [T(\bar{e}_1)\ T(\bar{e}_2) \cdots T(\bar{e}_n)]$ will be an $n \times n$ matrix.

$$T(\bar{e}_j) = 3\bar{e}_j = 3\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ position}$$

$j = 1, \dots, n$

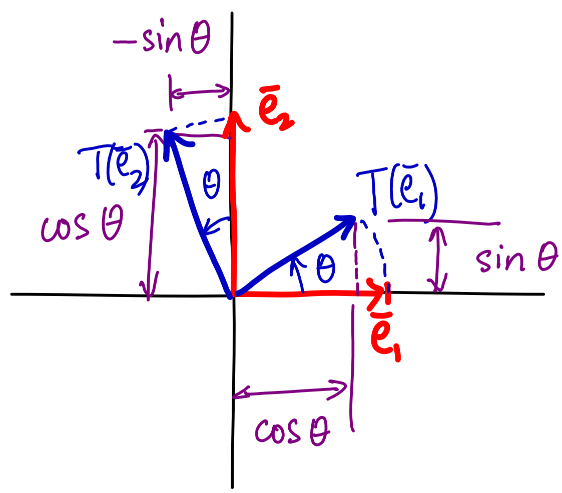
$$A = [3\bar{e}_1\ 3\bar{e}_2 \cdots 3\bar{e}_n] = \begin{bmatrix} 3 & 0 & 0 & \cdots & 0 \\ 0 & 3 & \ddots & & \vdots \\ \vdots & 0 & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 3 \end{bmatrix} = 3I_n.$$

$n \times n$ identity matrix

Geometric LTs of $\mathbb{R}^2$ (Find A s.t. $T(\bar{x}) = A\bar{x}$ in each case).

(1) Rotation (rotate counterclockwise (CCW) by θ degrees).

$$T(\bar{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad T(\bar{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



So, $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

e.g., $\theta = 60^\circ$, $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

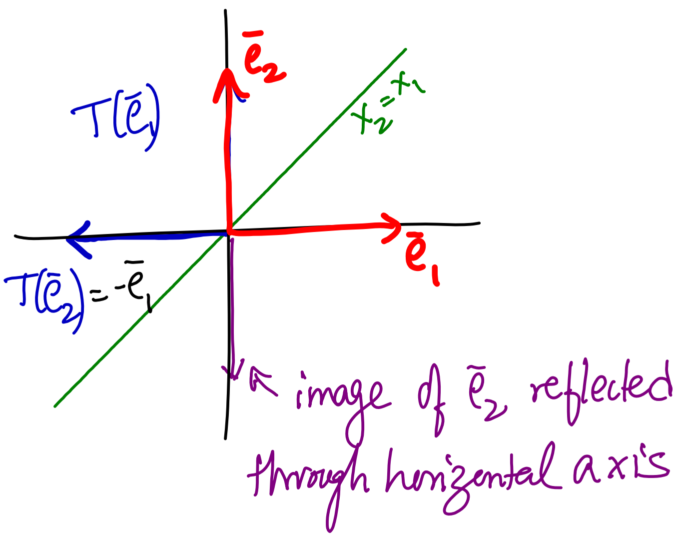
In the "Mangle the Coug" web page, input $A = \begin{bmatrix} 1/2 & -87/100 \\ 87/100 & 1/2 \end{bmatrix}$ (as it only takes fractions).

② <u>Reflections</u>

(Prob 8, Pg 90). Find A of LT: $x \mapsto A\bar{x}$ which reflects on the x-axis and then reflects through the line $x_2 = x_1$.

$T(\bar{e}_1) = \bar{e}_2$ and $T(\bar{e}_2) = -\bar{e}_1$

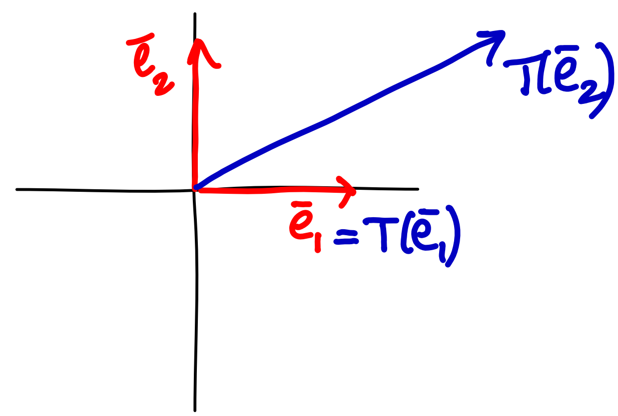
$$A = [\bar{e}_2 \ -\bar{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



③ <u>Shear</u> T is a horizontal shear transformation that leaves $\bar{e}_1$ unchanged, and map $\bar{e}_2$ to $\bar{e}_2 + 3\bar{e}_1$.

$$T(\bar{e}_2) = \bar{e}_2 + 3\bar{e}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A = [T(\bar{e}_1) \ T(\bar{e}_2)] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

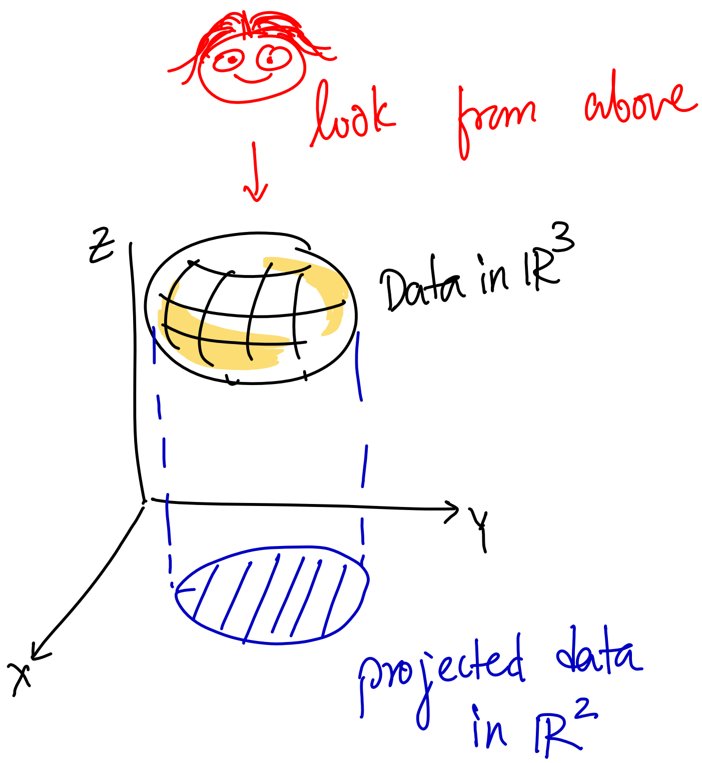


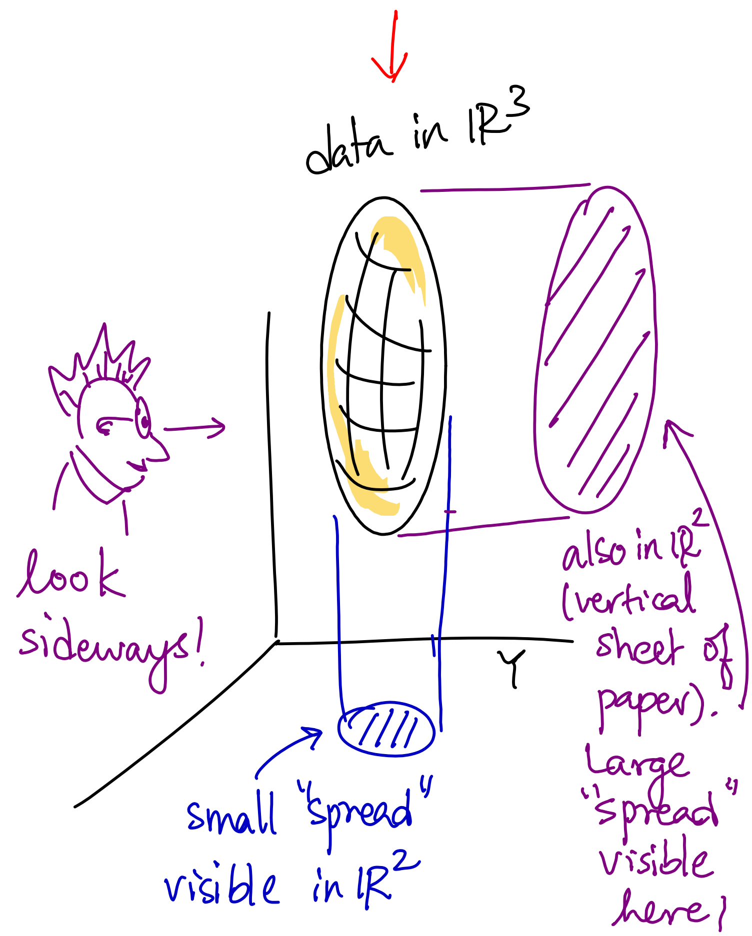
A vertical shear LT will leave $\bar{e}_2$ the same, and transform $\bar{e}_1$ to $\bar{e}_1 + c\bar{e}_2$ for some scalar c.

④ Projection from $\mathbb{R}^n$ to $\mathbb{R}^2$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$ e.g., $\bar{x}$ is part of high-dimensional data.

To visualize better, we could project $\bar{x}$ to $\mathbb{R}^2$.





One such projection $T: \mathbb{R}^n \to \mathbb{R}^2$ is given by

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ignore higher (≥ 3) dimensions

$$A = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) & \cdots & T(\bar{e}_n) \end{bmatrix}$$

$$T(\bar{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T(\bar{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T(\bar{e}_j) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for } j \geq 3.$$

$$\bar{e}_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \leftarrow j^{th} \text{ position} \\ \\ \\ \\ \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad 2 \times n \text{ matrix}$$