MATH230 – Lecture 12 (02/17/2011)

Onto and one-to-one transformations (functions or mappings)

Def $T: \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ (domain) **onto** $\mathbb{R}^m$ (codomain) if

need not be an LT

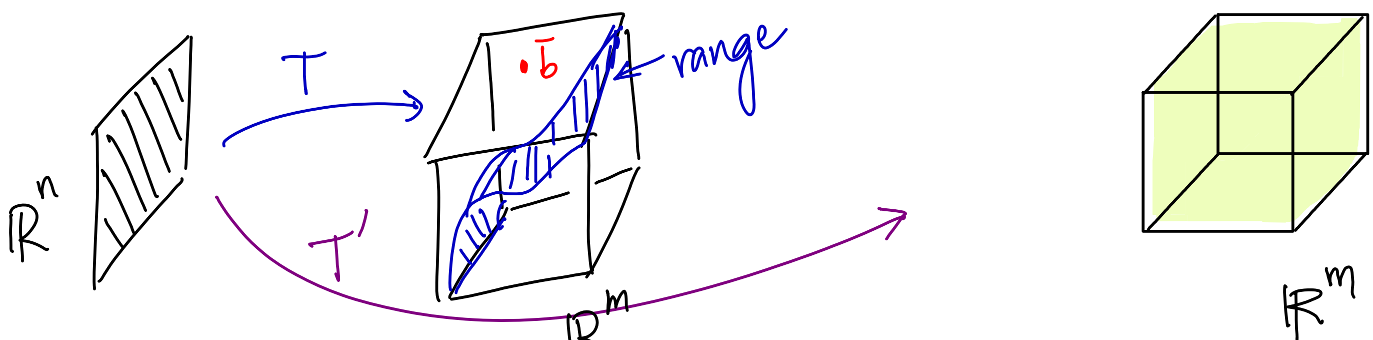
every $\bar{b} \in \mathbb{R}^m$ is the image under T of **at least** one $\bar{x} \in \mathbb{R}^n$. (i.e., $T(\bar{x}) = \bar{b}$ for at least one $\bar{x}$)

1 or more

$T: \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each $\bar{b} \in \mathbb{R}^m$ is the image under T of **at most** one $\bar{x} \in \mathbb{R}^n$.

1 or none

($T(\bar{x}) = \bar{b}$ for at most one $\bar{x}$)



domain

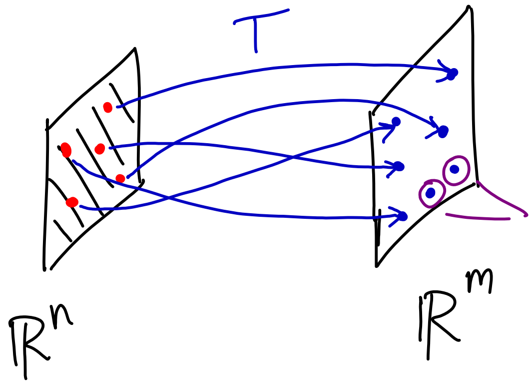
codomain

T is not onto, as the range is strictly smaller than the codomain.

(Not every $\bar{b} \in \mathbb{R}^m$ is $T(\bar{x})$ for some $\bar{x} \in \mathbb{R}^n$)

range = codomain

So, T' is onto.



T is one-to-one.

are not images of any $\bar{x} \in \mathbb{R}^n$

LTs that are onto or 1-to-1

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be an LT. So, $T(\bar{x}) = A\bar{x}$ for some $A \in \mathbb{R}^{m \times n}$.

Theorem 12 DL-LAA (pg 89)

1. T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if and only if columns of A **span** $\mathbb{R}^m$. $\}$ $A\bar{x} = \bar{b}$ is consistent for all $\bar{b} \in \mathbb{R}^m$.
2. T is **one-to-one** if and only if **columns** of A are linearly independent **(LI)**.

   $A\bar{x} = \bar{b}$ has unique solution or is inconsistent.

Sketch of proof

1. Columns of A span $\mathbb{R}^m \Rightarrow$ every row of A has a pivot, so $A\bar{x} = \bar{b}$ is consistent for every $\bar{b} \in \mathbb{R}^m$. So, there is at least one $\bar{x}$ such that $T(\bar{x}) = A\bar{x} = \bar{b}$.

2. Columns of A are LI $\Rightarrow$ every column has a pivot, or that there are **no free variables.** So $A\bar{x}=\bar{b}$ has at most one solution.

$$\underbrace{\left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{array}\right]}_{A \quad \bar{b}} \longrightarrow \text{inconsistent system.}$$

For LTs $\begin{cases} \textbf{pivot in every row} \Rightarrow \textbf{onto} \\ \textbf{pivot in every column} \Rightarrow \textbf{1-to-1} \end{cases}$

Prob 29, pg 91

$T: \mathbb{R}^3 \to \mathbb{R}^4$ is a 1-to-1 LT. Describe all possible echelon forms of the matrix of this LT.

$T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{4\times 3}$. T is 1-to-1 means every column of A has a pivot.

$$A = \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$ is the only echelon form possible.

<u>Prob 30, pg 91</u> $T: \mathbb{R}^4 \to \mathbb{R}^3$ is an onto LT. Describe all possible echelon forms of A such that $T(\bar{x}) = A\bar{x}$.

A is a 3×4 matrix, and should have a pivot in every row.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

are the possible echelon forms.

In general, if $T: \mathbb{R}^n \to \mathbb{R}^m$ is an LT, $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{m\times n}$.

* T is onto $\Rightarrow$ $m \le n$ → need pivot in every row; cannot have more rows than columns.
* T is one-to-one $\Rightarrow$ $m \ge n$ → need pivot in every column; cannot have more columns than rows
* T is one-to-one AND onto $\Rightarrow$ $m = n$

pivot in every row $\Rightarrow$ consistent for all $\bar{b}$
pivot in every column $\Rightarrow$ no free vars.

} Has a unique solution for each $\bar{b}$.

Prob 23, pg 91 (TRUE/FALSE)

(d) A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if every vector $\bar{x} \in \mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.

FALSE. We need every vector $\bar{b}$ in $\mathbb{R}^m$ to be mapped from ~~some~~ vector $\bar{x} \in \mathbb{R}^n$, not the other way around.

(e) If A is a 3×2 matrix, then the transformation $\bar{x} \mapsto A\bar{x}$ cannot be 1-to-1.

FALSE. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ gives a 1-to-1 map.

24(e) A is 3×2, then transformation $\bar{x} \mapsto A\bar{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

TRUE. $m > n$ here, so we cannot have a pivot in each row.

Applications (Section 1.10)

Difference Equations

Prob 10, pg 101

10. In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 3% of the suburban population moves into the city. In 2000, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where $\mathbf{x}_0$ is the initial population in 2000. Then estimate the population in the city and in the suburbs two years later, in 2002.

Let c_k = city population in year k
s_k = suburbs population in year k

$k=0$ at 2000, $k=1$ at 2001, and so on.

$\bar{x}_k = \begin{bmatrix} c_k \\ s_k \end{bmatrix}$ vector of populations in year k.

$c_0 = 800,000$, $s_0 = 500,000$ (starting populations).

Hence $\bar{x}_0 = \begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix}$.

$$c_{k+1} = \underbrace{(1-0.07)}_{\text{fraction of people who stayed put in the city}} c_k + \underbrace{0.03}_{\text{fraction of people from suburbs who moved in}} s_k$$

$$s_{k+1} = (1-0.03)s_k + 0.07c_k$$

similar arguments for suburbs

$$\bar{x}_{k+1} = \begin{bmatrix} c_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{matrix} 0.93c_k + 0.03s_k \\ 0.07c_k + 0.97s_k \end{matrix} = \underbrace{\begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix}}_{M \text{ migration matrix}} \begin{bmatrix} c_k \\ s_k \end{bmatrix}$$

$$\bar{x}_{k+1} = M\bar{x}_k \quad \text{for} \quad M = \begin{bmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{bmatrix}$$

add to 1   add to 1.

each column adds to 1.

To find populations in 2002, we find $\bar{x}_1$ (populations in 2001), and then $\bar{x}_2$.

(from MATLAB)

$$\bar{x}_1 = M\bar{x}_0 = \begin{bmatrix} 759000 \\ 541000 \end{bmatrix}; \quad \bar{x}_2 = M\bar{x}_1 = \begin{bmatrix} 722100 \\ 577900 \end{bmatrix}.$$