MATH230 – Lecture 13 (02/22/2011)

Matrix Operations (Section 2.1)

$$A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n] = \left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] \rightarrow i^{th} \text{ row}$$

$m \times n$

$\bar{a}_j \in \mathbb{R}^m$

j^{th} column

A_{ij} is used sometimes in place of a_{ij}.

Notation: uppercase letters stand for matrices,
e.g., $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n \times n}$ etc.

Matrix Addition

$A+B$ is defined when A and B have the same size. If $C = A+B$, then

$$c_{ij} = a_{ij} + b_{ij} \quad (\text{or } C_{ij} = A_{ij} + B_{ij}).$$

(add corresponding entries)

Scalar multiplication

$B = rA$ for scalar r, then

$$b_{ij} = r \cdot a_{ij}$$

(multiply each entry by scalar r).

Properties of matrix addition & scalar multiplication

(Theorem 1, pg 108).

$A, B, C \in \mathbb{R}^{m\times n}$, $r, s \in \mathbb{R}$

(a) $A + B = B + A$

(b) $(A+B)+C = A + (B+C)$

(c) $A + 0 = A$ mxn zeromatrix

(d) $r(A+B) = rA + rB$

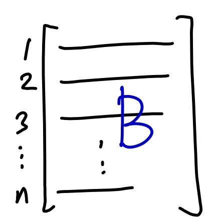
(e) $(r+s)A = rA + sA$

(f) $r(sA) = (rs)A$

Matrix multiplication

We have seen $A\bar{x}$ for mxn matrix A and n-vector $\bar{x}$, or an $n \times 1$ matrix $\bar{x}$.

Def $C = AB$ is defined when A is $m \times n$ and B is $n \times p$, i.e., when # columns in A = # rows in B.

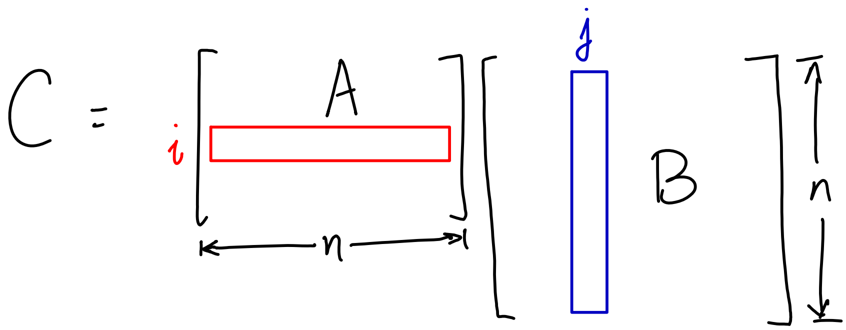
$A = [\bar{a}_1 \cdots \bar{a}_n]$ $\qquad$ $B = [\bar{b}_1 \cdots \bar{b}_p] \rightarrow$ 

$\bar{a}_j \in \mathbb{R}^m$ $\qquad$ $\bar{b}_i \in \mathbb{R}^n$

need to be same.

$$C = AB = A[\bar{b}_1 \cdots \bar{b}_p], \quad \bar{b}_i \in \mathbb{R}^n$$

$$= [A\bar{b}_1 \; A\bar{b}_2 \cdots A\bar{b}_p]$$



$$c_{ij} = \boxed{a_{i1} \; a_{i2} \cdots \cdots a_{in}} \boxed{\begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{matrix}} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}}$$

Example

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 4 \end{bmatrix}$$

2×2 $\qquad$ 2×3

$C = AB$ is defined, but BA is not defined.

$$C = \left[\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} \boxed{3} & 4 & -4 \\ 3 & \boxed{9} & 0 \end{bmatrix}$$

$2 \times 1 + (-1) \times (-1)$

$3 \times 3 + 0 \times 2$

$C = AB$ for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ will have size $m \times p$, i.e., $C \in \mathbb{R}^{m \times p}$.

Prob 10, pg 116

$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Is $AB = AC$?

What about $AB \stackrel{?}{=} BA$?

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 2\times 8 - 3\times 5 & 2\times 4 - 3\times 5 \\ -4\times 8 + 6\times 5 & -4\times 4 + 6\times 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2\times 5 - 3\times 3 & 2\times -2 - 3\times 1 \\ -4\times 5 + 6\times 3 & -4\times -2 + 6\times 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

So, $AB = AC$.

$$BA = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 8\times 2 + 4\times -4 & 8\times -3 + 4\times 6 \\ 5\times 2 - 5\times 4 & 5\times -3 + 5\times 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -10 & 15 \end{bmatrix}$$

$AB \neq BA$ here.

In general, $AB \neq BA$. In fact, BA may not even be defined. If $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 3}$, then AB is defined, but BA is not.

Properties of matrix multiplication

(Theorem 2, pg 113)

* AB is not typically equal to BA (BA may not even be defined)

(a) $A(BC) = (AB)C$ (associativity)

(b) $A(B+C) = AB+AC$ (left distributive property)

(c) $(B+C)A = BA+CA$ (right distributive property)

(d) $r(AB) = (rA)B = A(rB)$

(e) $I_m A = A = AI_n$ (identity)

I_m: $m\times m$ identity, I_n: $n\times n$ identity matrix

Proof of (b). $A(B+C) = AB+AC$

$A_{m\times n}$

$B_{n\times p}$

$C_{n\times p}$

$B = [\bar{b}_1\ \bar{b}_2 \cdots \bar{b}_p]$

$C = [\bar{c}_1\ \bar{c}_2 \cdots \bar{c}_p]$

$$A(B+C) = A\left([\bar{b}_1 \bar{b}_2 \cdots \bar{b}_p] + [\bar{c}_1\ \bar{c}_2 \cdots \bar{c}_p]\right)$$

$$= A\left[(\bar{b}_1+\bar{c}_1)\ (\bar{b}_2+\bar{c}_2) \cdots (\bar{b}_p+\bar{c}_p)\right]$$

from matrix addition of B & C

$$= \left[A(\bar{b}_1+\bar{c}_1)\ A(\bar{b}_2+\bar{c}_2) \cdots A(\bar{b}_p+\bar{c}_p)\right]$$

$$= [A\bar{b}_1 + A\bar{c}_1 \quad A\bar{b}_2 + A\bar{c}_2 \quad \cdots \quad A\bar{b}_p + A\bar{c}_p]$$ from matrix-vector multiplication

$$= [A\bar{b}_1 \; A\bar{b}_2 \cdots A\bar{b}_p] + [A\bar{c}_1 \; A\bar{c}_2 \cdots A\bar{c}_p]$$

$$= AB + AC$$

Transpose of a matrix A (A^T)

T in superscript

Interchange rows and columns of A to get A^T.

e.g. $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -3 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 4 \\ 1 & -3 \\ 0 & 6 \end{bmatrix}$

2×3 $\qquad$ 3×2.

If $A \in \mathbb{R}^{m \times n}$, then $A^T \in \mathbb{R}^{n \times m}$.

Properties of matrix transposes

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$

transpose of product = product of transposes in reverse order

Prob 10, pg 116 (contd...)

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}.$$ Verify that $(AB)^T = B^T A^T$.

$$AB = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$ (from before - see page 13-4)

$$\Rightarrow (AB)^T = \begin{bmatrix} 1 & -2 \\ -7 & 14 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 8 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -7 & 14 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 8 & 5 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 0 & 15 \end{bmatrix} = (BA)^T$$

Power of a matrix

If $A \in \mathbb{R}^{n \times n}$, then $A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$

e.g.,

When $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 16 & -24 \\ -32 & 48 \end{bmatrix}$.

We will do an example that motivates the definition of the "inverse" of a matrix.

Prob 17, pg 117

If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$, and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, find B.

Since A is 2×2, and AB 2×3, B must be a 2×3 matrix.

Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$.

$$AB = \overset{A}{\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}} \begin{array}{l} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \\ \ \ \bar{x} \end{array} = \begin{bmatrix} b_{11}-2b_{21} & b_{12}-2b_{22} & b_{13}-2b_{23} \\ -2b_{11}+5b_{21} & -2b_{12}+5b_{22} & -2b_{13}+5b_{23} \end{bmatrix}$$

$$= \begin{array}{l} \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} \\ \ \ \bar{c} \end{array}$$

We can find $\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$ by solving the system of linear equations given as $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Similar calculations could give us $\bar{b}_2 = \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix}$ and $\bar{b}_3 = \begin{bmatrix} b_{31} \\ b_{32} \end{bmatrix}$. Notice that the coefficient matrix A is the same in all three systems here!