MATH 230 – Lecture 14 (02/24/2011)

$$AB = \overset{A}{\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

$$\qquad\qquad\qquad \bar{b}_1 \quad \bar{b}_2 \qquad\qquad\qquad \bar{c}_1 \quad \bar{c}_2$$

$A\bar{b}_1 = \bar{c}_1$ is a system of two equations in two variables.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]$$

$$A \quad \bar{c}_1$$

Similarly, we can solve $A\bar{b}_2 = \bar{c}_2$ and $A\bar{b}_3 = \bar{c}_3$. Notice that the EROs are the same as in the case of $A\bar{b}_1 = \bar{c}_1$. We could "club" the three augmented matrices, and do the same EROs in one go.

$$\left[\begin{array}{cc|ccc} 1 & -2 & -1 & 2 & -1 \\ -2 & 5 & 6 & -9 & 3 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|ccc} 1 & -2 & -1 & 2 & -1 \\ 0 & 1 & 4 & -5 & 1 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{cc|ccc} 1 & 0 & 7 & -8 & 1 \\ 0 & 1 & 4 & -5 & 1 \end{array}\right]$$

$$A \quad \bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underbrace{\qquad\qquad}_{B}$$

Hence $B = \begin{bmatrix} 7 & -8 & 1 \\ 4 & -5 & 1 \end{bmatrix}$.

With $a, b \in \mathbb{R}$, let $ab = c$. If you are given a and c, to find b, we do, assuming $a \neq 0$,

$$a \cdot b = c$$
$$b = \frac{1}{a} c = a^{-1} \cdot c$$

The previous example had

$$AB = C \quad (C \text{ being the product } AB \text{ given}).$$

Can we write $B = A^{-1}C$?

We define the inverse of a matrix.

<u>(Section 2.2)</u>

If $A \in \mathbb{R}^{n \times n}$, and if there is a matrix $B \in \mathbb{R}^{n \times n}$ such that $AB = BA = I_n$, then B is called the **inverse** of A, and is denoted as $B = A^{-1}$.

If A^{-1} exists, we say that A is **invertible**.

e.g., $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

Check: $AA^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{-1}A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Theorem 4, DL-LAA pg 119

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A^{-1} exists when $ad-bc \neq 0$.

If $ad-bc \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The quantity $ad-bc$ is called the **determinant** of A. It holds in general (not just for 2x2 matrices) that A is invertible when its determinant is not zero.

$ax = b, a \neq 0 \Rightarrow x = \frac{1}{a}b = a^{-1}b$

Theorem 5, DL-LAA page 120

If A is invertible, then $A\bar{x} = \bar{b}$ has a unique solution for any b, given by $\bar{x} = A^{-1}\bar{b}$.

In the previous example ($AB=C$, find B), we could have just calculated B as $A^{-1}C$.

$$A^{-1}C = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -8 & 1 \\ 4 & -5 & 1 \end{bmatrix}.$$

Review for midterm

Hw6. Prob 2

Project from $\mathbb{R}^4$ to $\mathbb{R}^2$ by ignoring 3^{rd} & 4^{th} coordinates, then rotate ccw by 45°.

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \cancel{0} \\ \cancel{0} \end{bmatrix} \xrightarrow[\text{to } \mathbb{R}^2]{\text{project}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow[\text{ccw } 45^\circ]{\text{rotate}} \begin{bmatrix} \cos 45 \\ \sin 45 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \cancel{0} \\ \cancel{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -\sin 45 \\ \cos 45 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\bar{e}_3 = \begin{bmatrix} 0 \\ 0 \\ \cancel{1} \\ \cancel{0} \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ stays same post rotation}$$

$$\bar{e}_4 = \begin{bmatrix} 0 \\ 0 \\ \cancel{0} \\ \cancel{1} \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ stays same post rotation}$$

The matrix of LT $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix}$.

Practice Midterm

#6 (Prob 14, DL-LAA page 103)

$\left\{ \begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ a+2 \end{bmatrix} \right\}$ For what a are these vectors LI?

$A = \begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix}$. A must have a pivot in each column for its columns to be LI.

$$\begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix} \xrightarrow{R_2 - aR_1} \begin{bmatrix} 1 & a \\ 0 & a+2-a^2 \end{bmatrix} \neq 0$$

$a+2-a^2 = 0$ gives $a^2 - a - 2 = 0$, i.e., $(a-2)(a+1) = 0$

So $a+2-a^2=0$ for $a = 2, -1$. Hence the given two vectors are LI for $\underbrace{a \in \mathbb{R} / \{2, -1\}}_{\text{all real values except } 2,-1}$.

all real values except 2, -1.

or just say $a \neq 2, -1$.

#8 True/False

(a) FALSE.

A does not have a pivot in every row. Since A is square, it does not have a pivot in every column, so there exist(s) free variable(s).

(b) FALSE.

We get the solutions to $A\bar{x}=\bar{b}$ by adding some vector to that of $A\bar{x}=\bar{0}$, not necessarily $\bar{b}$.

(see answers to Prob #1. If A is 3x4, then $\bar{b} \in \mathbb{R}^3$, but the vector you add will be in $\mathbb{R}^4$).

(c) FALSE. $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ is LD, but has 1 vector with 2 entries.

Only the converse is true, i.e., if there are more vectors than entries in vectors, the set is LD.

(d) FALSE. $\bar{x} \mapsto A\bar{x}$ where $A=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A has a pivot in every column, but not in every row.

#5. $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T(\bar{e}_1) = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}, \; T(\bar{e}_2) = \begin{bmatrix} 3 \\ k \\ 0 \end{bmatrix}.$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix} \xrightarrow[R_3 - hR_2]{R_1 - 2R_2} \begin{bmatrix} 0 & 3-2k \\ 1 & k \\ 0 & -hk \end{bmatrix} \xrightarrow{R_1 \rightleftharpoons R_2} \begin{bmatrix} 1 & k \\ 0 & 3-2k \\ 0 & -hk \end{bmatrix}$$

Both columns of A have a pivot if either $3-2k \neq 0$ or $-hk \neq 0$, i.e, $k \neq \frac{3}{2}$ or $h \neq 0, k \neq 0$. So T is 1-to-1 for $h \in \mathbb{R}, k \in \mathbb{R}/\{3/2\}$ and $h, k \in \mathbb{R}/\{0\}$.

T cannot be onto, as A cannot have a pivot in every row, i.e, T is onto for <u>NO</u> h & k.

#3. If $A_{3\times4}$ has a pivot in every row, then $\bar{b}$ is in span of columns of A for all $\bar{b} \in \mathbb{R}^3$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{to convert 0's to nonzeros}]{\text{Use EROs}}$$