MATH230 – Lecture 16 (03/03/2011)

Offer to possibly get A in MATH 230:

* If you score 95 or more in the Final, your final score will replace your mid-term score.
* If you score ≥ 90 in the Final, the weights for midterm and final would be changed to 15% and 40%, respectively.

Inverse of an $n\times n$ matrix

If $A \in \mathbb{R}^{n\times n}$, and if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$, then $A^{-1} = B$. (A is invertible, B is the inverse of A; A is the inverse of B).

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is Invertible if $\det A = ad - bc \neq 0$ (determinant). Then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 4, DL-LAA page 119

If $A \in \mathbb{R}^{n\times n}$ is invertible, then $A\bar{x}=\bar{b}$ has a unique solution for any $\bar{b} \in \mathbb{R}^n$ given by
$\bar{x} = A^{-1}\bar{b}$.   Theorem 5, DL-LAA page 120.

<u>Prob 6, pg 126</u>   Use matrix inverse to solve the system.

$$8x_1 + 5x_2 = -9$$
$$-7x_1 - 5x_2 = 11$$

$A\bar{x} = \bar{b}$ where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$. $\det A = 8\times -5 - (-7)\times 5$
$= -40+35 = -5 \neq 0$

So, A^{-1} exists. $A^{-1} = \dfrac{1}{-5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}$.

Hence the unique solution is $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\bar{b} = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

In MATLAB, the function **inv()** gives the inverse of a matrix.

e.g.,   >> Ainv = inv(A)

Properties of matrix inverses ($A, B \in \mathbb{R}^{n\times n}$, invertible)

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ → inverse of product = product of inverses in reverse order
3. $(A^T)^{-1} = (A^{-1})^T$

Proof sketch

2. $(AB)^{-1} = B^{-1}A^{-1}$

We want $C \in \mathbb{R}^{n\times n}$ such that

$C(AB) = (AB)C = I$.

Can just verify that $C = B^{-1}A^{-1}$ works.

$$(B^{-1}\underbrace{A^{-1})(A}B) = B^{-1}(A^{-1}A)B = B^{-1}\underbrace{I B} = B^{-1}B = I$$

$$(A\underbrace{B)(B^{-1}}A^{-1}) = A(BB^{-1})A^{-1} = \underbrace{AI}A^{-1} = AA^{-1} = I$$

3. $(A^T)^{-1} = (A^{-1})^T$. We want C such that $CA^T = A^TC = I$.

$C = (A^{-1})^T$ works!

$$\underbrace{(A^{-1})^T A^T = (AA^{-1})^T} = I^T = I$$

$$\underbrace{A^T (A^{-1})^T = (A^{-1}A)^T} = I^T = I.$$

as $(AB)^T = B^T A^T$

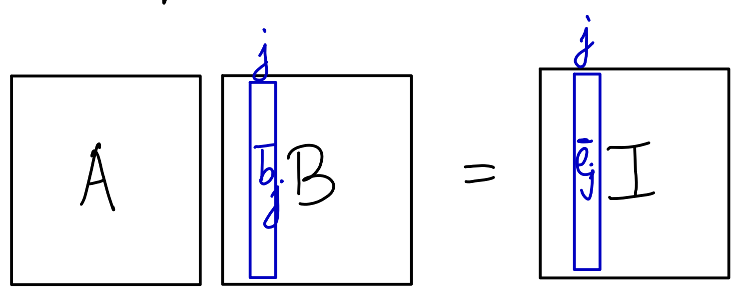
How to invert an $n \times n$ matrix? $(n \geq 3)$

Say we want to find $B = A^{-1}$. We have $AB = I$.

We know how to solve $A\bar{x} = \bar{b}$

$$[A \mid \bar{b}] \xrightarrow{\text{EROs}} \text{rref}([A \mid \bar{b}]).$$

We could just solve for each column of B as a system of n equations in n variables.



For $\bar{b}_j$ (j^{th} column of B), we solve

$$A\bar{b}_j = \bar{e}_j \longrightarrow \text{the } j^{th} \text{ unit vector.}$$

The augmented matrix for this system is $[A \mid \bar{e}_j]$.
We could combine the augmented matrices for all j, and look at $[A \mid I]$, and perform EROs on it.

We'll get $[A \mid I] \xrightarrow{\text{EROs}} [I \mid A^{-1}]$ if A^{-1} exists.

Therem 7 DL-LAA pg 123

$A \in \mathbb{R}^{n\times n}$ is invertible if and only if it is row-equivalent to I_n (i.e, you can reduce A to I_n using EROs). In this case, the same EROs reduce I_n to A^{-1}. So, $\text{rref}([A|I_n]) = [I_n|A^{-1}]$

If we do not get I_n in place of A, then A is not invertible.

Prob 32 pg 127

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$ Find A^{-1} if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-4R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 2 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$$

A is not invertible, as $A \not\sim I_3$.

↳ not row-equivalent

"$\sim$": row-equivalent

Prob 14 pg 126

$B, C \in \mathbb{R}^{m\times n}$, D is invertible, and $(B-C)D = O$.

Show that $B = C$.

Since $(B-C)D$ is defined, D has n row. But D is invertible, so D is $n\times n$.

$$\underbrace{(B-C)}_{m\times n}\underbrace{D}_{n\times n} = \underbrace{O}_{m\times n}$$

D^{-1} exists. Multiply the above equation on the right by D^{-1}.

$$(B-C)DD^{-1} = OD^{-1}$$

$$\Rightarrow (B-C)(DD^{-1}) = OD^{-1} = O$$

$$\Rightarrow (B-C)I = O, \text{ i.e., } B-C = O$$

$$\Rightarrow B = C.$$

Prob 18, pg 126

P is invertible, $A = PBP^{-1}$. Solve for B.

$$(A = PBP^{-1})\,P$$

$$AP = PB\underbrace{P^{-1}P} = PBI = PB$$

Multiply on left with P^{-1} to get

$$P^{-1}AP = \underbrace{P^{-1}P}B = IB = B.$$

So $\quad B = P^{-1}AP$

For $A, B \in \mathbb{R}^{n\times n}$, if there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that $A = P^{-1}BP$, then A and B are **similar**. Then $B = PAP^{-1}$.

Prob 20, pg 126

$A, B, X \in \mathbb{R}^{n\times n}$, A, X and $(A-AX)$ are invertible, and

$$(A-AX)^{-1} = X^{-1}B \quad \text{——— (1)}$$

(a) Is B invertible?

(b) Solve for X from (1).