MATH230 – Lecture 17 (03/08/2011)

Properties of invertible matrices

Prob 20, from Pg 126: A, X, and $(A-AX)$ are invertible. $A, B, X \in \mathbb{R}^{n\times n}$.

$$(A-AX)^{-1} = X^{-1}B \quad \text{———} \quad (1)$$

(a) Is B invertible?

(b) Solve for X from (1).

$$(A-AX)^{-1} = X^{-1}B$$

$$\Rightarrow X(A-AX)^{-1} = X(X^{-1}B) \qquad X \in \mathbb{R}^{n\times n}, \ (A-AX)^{-1} \text{ is also } n\times n$$

$$= IB = B \qquad \text{as } A(BC) = (AB)C$$

$$\Rightarrow B^{-1} = [X(A-AX)^{-1}]^{-1} = ((A-AX)^{-1})^{-1}X^{-1} \qquad \text{as } (AB)^{-1} = B^{-1}A^{-1}$$

$$= (A-AX)X^{-1} \qquad \text{as } (A^{-1})^{-1} = A$$

So B^{-1} exists, showing that B is invertible.

(b) $(A-AX)^{-1} = X^{-1}B \rightsquigarrow \frac{1}{a-ax} = \frac{b}{x}$

Taking inverse on both sides gives

$$A-AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$$ as $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^{-1})^{-1} = A$.

$$\Rightarrow A = AX + B^{-1}X = (A+B^{-1})X$$

Is $(A+B^{-1})$ invertible? YES, as A and X are invertible

Caution! A, B invertible $\not\Rightarrow$ $A+B$ is invertible.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, but $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\det A = 1$ $\det B = 1$ $\det(A+B) = 0$

$$\left(A = (A+B^{-1})X\right)X^{-1} \Rightarrow \left[AX^{-1} = A+B^{-1}\right]^{-1} \Rightarrow (A+B^{-1})^{-1} = XA^{-1}$$

$$(A+B^{-1})^{-1}\left[A = (A+B^{-1})X\right] \Rightarrow$$

$$(A+B^{-1})^{-1}A = IX = X, \quad \text{i.e., } X = (A+B^{-1})^{-1}A.$$

Inverse of $n \times n$ Matrices

$$[A \mid I] \xrightarrow{\text{EROs}} [I \mid A^{-1}] \quad \text{if } A \text{ is invertible.}$$

Prob 32, pg 127 Find inverse of $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -5 \end{bmatrix}$

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$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-4R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -3 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 2 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 10 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2-R_3,]{R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & 4 & -1 \\ 0 & 1 & 0 & -14 & 3 & -1 \\ 0 & 0 & 1 & -10 & 2 & -1 \end{array}\right]$$

and $R_3 \times (-1)$

Prob 22, pg 126 Explain why columns of $A \in \mathbb{R}^{n \times n}$ span $\mathbb{R}^n$ when A is invertible.

Since A is invertible, it is row-equivalent to I_n. Hence, it has a pivot in every column and every row. Since there is a pivot in every row, columns of A span $\mathbb{R}^n$.

Prob 33 Pg 127 Find inverses of $\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$.

Guess the inverse of $A=\begin{bmatrix}1&0&\cdots&0\\1&1&0&\vdots\\\vdots&1&\ddots&0\\1&1&\cdots&1\end{bmatrix}$.

$$\left(\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)^{-1}=\frac{1}{1}\begin{bmatrix}1&0\\-1&1\end{bmatrix}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\1&1&0&0&1&0\\1&1&1&0&0&1\end{array}\right]\xrightarrow[R_3-R_1]{R_2-R_1}\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&1&1&-1&0&1\end{array}\right]\xrightarrow{R_3-R_2}\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&0&1&0&-1&1\end{array}\right]$$

Guess: $A^{-1}=\begin{bmatrix}1&&&\mathbf{0}\\-1&1&&\\&-1&\ddots&\\\mathbf{0}&&-1&1\end{bmatrix}$?

$$AA^{-1}=\begin{array}{c}\begin{array}{cccccc}&&j&j+1&&\end{array}\\\begin{bmatrix}1&0&0&\cdots&\cdots&0\\1&1&0&&&\vdots\\\vdots&\vdots&1&&\ddots&0\\1&1&1&\cdots&\cdots&1\end{bmatrix}\end{array}\begin{array}{c}\begin{array}{ccc}&j&\end{array}\\\begin{bmatrix}1&&&\\-1&1&&\\&-1&\ddots&\\&&-1&1\end{bmatrix}\end{array}$$

$\begin{bmatrix}0\\\vdots\\0\\1\\-1\\0\\\vdots\\0\end{bmatrix}$ ← j^{th} position for $1\le j\le n-1$

$$(1)\begin{bmatrix}0\\\vdots\\1\\1\\\vdots\\1\end{bmatrix}\begin{array}{l}\\\\\leftarrow j\\\\\\\\\end{array}+(-1)\begin{bmatrix}0\\\vdots\\0\\1\\\vdots\\1\end{bmatrix}\begin{array}{l}\\\\\\\leftarrow j{+}1\\\\\\\end{array}=\begin{bmatrix}0\\\vdots\\1\\0\\\vdots\\0\end{bmatrix}=\bar{e}_j \text{ the } j^{th} \text{ unit vector.}$$

So $AA^{-1}=I_n$. Can check that $A^{-1}A=I_n$ as well.

Invertible Matrix Theorem (IMT) (Section 2.3)

Theorem 8, DL-LAA Page 129.

The following statements are equivalent for $A \in \mathbb{R}^{n \times n}$.

a. A is invertible. → A^{-1} is invertible as well.

b. A is row equivalent to I_n.

c. A has n pivots. → pivot in every row & pivot in every column

d. $A\bar{x} = \bar{0}$ has only the trivial solution.

e. Columns of A are LI.

f. The LT $\bar{x} \mapsto A\bar{x}$ is 1-to-1.

g. $A\bar{x} = \bar{b}$ has a unique solution for every $\bar{b} \in \mathbb{R}^n$.

h. Columns of A Span $\mathbb{R}^n$.

i. $\bar{x} \mapsto A\bar{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

j. $\exists\ C \in \mathbb{R}^{n \times n}$ such that $CA = I_n$.

"There exists"

k. $\exists\ D \in \mathbb{R}^{n \times n}$ such that $AD = I_n$

l. A^T is invertible.

Prob 17, Pg 133 If $A \in \mathbb{R}^{n\times n}$ is invertible, then columns of A^{-1} are LI. Explain why.

could skip

Since A is invertible, A^{-1} is invertible. Hence A^{-1} is row equivalent to I_n, i.e., it has a pivot in every column. Hence columns of A^{-1} are LI.

Prob 20, Pg 133 $E, F \in \mathbb{R}^{n\times n}$ satisfy $EF = I$. Explain why E and F commute.

E and F commute means $EF = FE$

Since $E, F \in \mathbb{R}^{n\times n}$ and $EF = I$, both E and F are invertible. Hence, from $EF = I$, we get

$$F = E^{-1}I = E^{-1} \quad \text{and} \quad E = F^{-1}$$

So $EF = FE = I$.