MATH230 - Lecture 18 (03/10/2011)

Prob 28 pg 133

Show that if AB is invertible, and A and B are square, then both A and B are invertible.

there exists

AB is invertible $\Rightarrow \exists\, C \in \mathbb{R}^{n\times n}$ such that

$$\underbrace{CA}B = I \Rightarrow (CA)B = I$$

$\Rightarrow \exists\, E \in \mathbb{R}^{n\times n}$ such that $EB = I$, where $E = CA$. Hence B is invertible.

AB is invertible $\Rightarrow \exists\, D \in \mathbb{R}^{n\times n}$ such that

$$A\underbrace{BD} = I$$

$\Rightarrow \exists\, F \in \mathbb{R}^{n\times n}$ such that $AF = I$, where $F = BD$. $\Rightarrow A$ is invertible.

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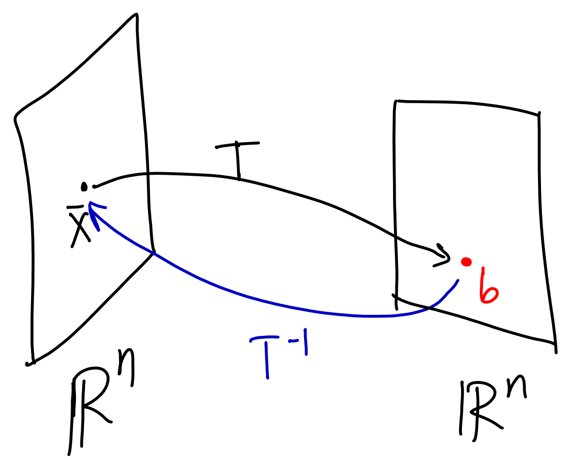
T is an LT which maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Show that T^{-1} exists, and maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Is T^{-1} 1-to-1?

T is an LT mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$.

$\Rightarrow$ $T(\bar{x}) = A\bar{x}$ for $A \in \mathbb{R}^{n \times n}$, with a pivot in every row. Hence A is invertible.

Detour T^{-1} is the inverse LT of T, i.e., if

$$T(\bar{x}) = \bar{b} \text{ then } T^{-1}(\bar{b}) = \bar{x}$$



$T(\bar{x}) = A\bar{x}$ so

$T^{-1}(A\bar{x}) = \bar{x}$. Hence

$T^{-1}(\bar{x}) = A^{-1}\bar{x}$, so that

$T^{-1}(A\bar{x}) = A^{-1}(A\bar{x}) = \bar{x}$

$$[A \mid I] \xrightarrow{\text{EROs}} [I \mid A^{-1}]$$

Since $A \sim I$, $A^{-1} \sim I$. row equivalent.

Hence A^{-1} has a pivot in every row and every column. Hence $T^{-1}(\bar{x})$ is also onto and 1-to-1.

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$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2).$

Show that T is invertible and find a formula for T^{-1}

T is invertible if $T(\bar{x}) = A\bar{x}$ and A is invertible.

Here, $T(\bar{x}) = A\bar{x}$, where $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$

$$A^{-1} = \frac{1}{(-5 \times -7 - 4 \times 9)} \begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$$

Since A is invertible, T is invertible, and

$$T^{-1}(\bar{x}) = A^{-1}\bar{x}.$$

Hence $T^{-1}(x_1, x_2) = (7x_1 + 9x_2, \ 4x_1 + 5x_2)$

Determinants (Section 3.1)

Recall, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A = ad - bc$. If $\det A \neq 0$,

then $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In general, A is invertible if det $A \neq 0$.

determines if A is invertible.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}$. When is A invertible?

what condition should a, b, c, d satisfy?

We need a pivot in every row and every column.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & b-12 \\ 0 & c-10 & d-15 \end{bmatrix} \xrightarrow{R_3 - \left(\frac{c-10}{a-8}\right)R_2}$$

$a - 8 \neq 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & a-8 & b-12 \\ 0 & 0 & (d-15) - \frac{(c-10)}{(a-8)}(b-12) \end{bmatrix}$$

$\neq 0$ to get three pivots

$$\Rightarrow \quad (a-8)(d-15) - (c-10)(b-12) \neq 0$$

$$ad - 8d - 15a + \cancel{120} - [bc - 10b - 12c + \cancel{120}] \neq 0$$

$$(ad-bc) - (8d-10b) + (12c-15a) \neq 0$$

$$\Rightarrow \quad 1(ad-bc) - 2(4d-5b) + 3(4c-5a) \neq 0$$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & a & b \\ 5 & c & d \end{bmatrix}$ We can write,

$$1 \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + (-1)\, 2 \det\left(\begin{bmatrix} 4 & b \\ 5 & d \end{bmatrix}\right) + 3 \det\left(\begin{bmatrix} 4 & a \\ 5 & c \end{bmatrix}\right)$$

This is the expression for $\det A$, expanding along row 1.

In general, for $A \in \mathbb{R}^{n \times n}$, $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n},$$

where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by removing Row 1 and column j from A.

expanding along Row 1.

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$$\begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix} = 8\begin{vmatrix} 0 & 3 \\ -2 & 5 \end{vmatrix} - 1\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix}$$

$|A| \to$ notation for $\det A$

$$= 8(0\times 5 - -2\times 3) - 1(4\times 5 - 3\times 3) + 6(4\times -2 - 3\times 0)$$

$$= 48 - 11 - 48 = -11.$$

We can expand along any row or any column of A to calculate $\det A$. Let $C_{ij} = (-1)^{i+j}\det A_{ij}$ be the (i,j)-cofactor of A.

Expanding along Column j,

$$\det A = a_{1j}C_{1j} + \dots + a_{nj}C_{nj},$$

and along Row i

$$\det A = a_{i1}C_{i1} + \dots + a_{in}C_{in}.$$

$$\begin{vmatrix} 8 & \boxed{1} & 6 \\ 4 & \boxed{0} & 3 \\ 3 & \boxed{-2} & 5 \end{vmatrix} = (-1)^{1+2}1\cdot\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} + \cancel{(-1)^{2+2}0\cdot\begin{vmatrix} 8 & 6 \\ 3 & 5 \end{vmatrix}} + (-1)^{2+3}2\cdot\begin{vmatrix} 8 & 6 \\ 4 & 3 \end{vmatrix}$$

$$= -11 + 0 + 0 = -11$$

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$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = (-1)^{4+1} \cdot 3 \cdot \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{vmatrix}$$

$$= (-3) \cdot (-1)^{3+4} \cdot 1 \cdot \begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 3\left\{ 3(-4\times 2 - 3\times 1) + 2(2\times 1 - {-4}\times 4) \right\} = 3(-33+36) = 9.$$

Theorem 2, DL-LAA pg 189

If $A \in \mathbb{R}^{n\times n}$ is triangular, then $\det A$ is the product of the entries in the diagonal.

$$\begin{bmatrix} \diagdown & 0 & 0 \\ * & \diagdown & 0 \\ * \; * & * \; * & \diagdown \end{bmatrix} \longrightarrow$$ lower triangular (all entries above diagonal are zero).

$$\begin{bmatrix} \diagdown & * \; * & * \\ 0 & \diagdown & * \; * \\ 0 \; 0 & 0 & \diagdown \end{bmatrix} \longrightarrow$$ upper triangular

$$A = \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{matrix}} & \begin{matrix} * \\ a_{22} \\ 0 \\ \vdots \end{matrix} & \begin{matrix} \\ * \\ \ddots \\ 0 \end{matrix} & \begin{matrix} * \\ \\ \\ a_{nn} \end{matrix} \end{bmatrix}$$

$$\det A = a_{11} \cdot \begin{vmatrix} a_{22} & * & * \\ 0 & a_{33} & * \\ \vdots & \ddots & \ddots \\ 0 & 0 & a_{nn} \end{vmatrix}$$

$$= a_{11} \cdot a_{22} \cdot \dots a_{nn}.$$

Properties of determinants

Example. $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow{R_2 + kR_1} \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix} = A'$

$\det A = 3\times 6 - 5\times 4 = -2$

$$\det A' = 3(6+4k) - 4(5+3k)$$

$$= -2 + \cancel{12k} - \cancel{12k}$$

$$= \det A.$$

In general, replacement EROs do not change $\det A$.