MATH 230 – Lecture 19 (03/22/2011)

Office Hours for tomorrow (Wed, Mar 23) are from noon - 2 pm.
Otherwise, I'll be in Neill 216 (come get me from there).

Determinants $A \in \mathbb{R}^{n \times n}$

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$ → expansion along Row i

where $C_{ij} = (-1)^{i+j} \det A_{ij}$ with $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ the submatrix of A obtained by deleting Row i and Column j.

Question: Can we use EROs to simplify determinant calculations?

Theorem 3, DL-LAA pg 192

(a) Replacement EROs do not change the determinant.

If $A \xrightarrow{R_i + kR_j} B$, $\quad (A, B \in \mathbb{R}^{n \times n})$

then $\det A = \det B$.

(b) $A \xrightarrow{R_i \rightleftarrows R_j} B$. Then $\det B = -\det A$.

(c) $A \xrightarrow[k \neq 0]{kR_i} B$. Then $\det B = k \cdot \det A$.

Using EROs to evaluate det A

Let U be the echelon form of A obtained using only replacement and exchange EROs. Then

$$\det A = \begin{cases} (-1)^r \times (\text{product of pivots in } U), & r = \#\text{ exchange EROs} \\ 0 \text{ if } A \text{ is not invertible.} \end{cases}$$

Remember, we **do not use any scaling EROs** to get U from A.

Prob 8 pg 199 Find determinant using row reduction.

$$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} \xrightarrow[R_4+3R_1]{R_3-2R_1} \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{vmatrix} \xrightarrow{R_3+R_2} \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \\ 0 & 2 & 4 & -10 \end{vmatrix} = 0$$

We can also combine row reduction and cofacter expansion to find determinants.

Prob 14 Pg 199

expand along column 3

$$\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} \xrightarrow{R_3+2R_1} \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -9 & 0 & 0 & 0 \\ 3 & -4 & 0 & 4 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix} = (-1)^{2+1} (-9) \begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix}$$

$$= 9\left(3 \times 4 - (-3)(-4)\right) = 0.$$

Note. $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\det A \neq 0$.

Theorem 5 DL-LAA pg 196

$$\det A^T = \det A$$

Proof (By induction).

Show for base case, i.e., $n=1$.
Assume result holds for $n=k$.
Show it holds for $n=k+1$.

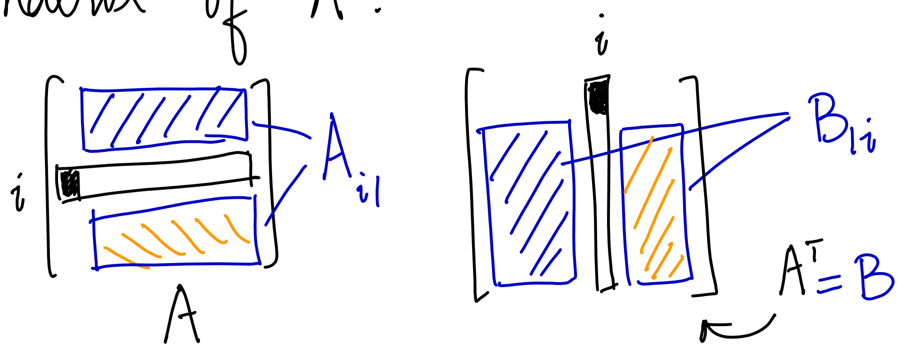
Let $A \in \mathbb{R}^{n \times n}$.

The result holds for $n=1$, as taking transpose of just a number gives back the same number.

Assume result holds for $n=k$, i.e., for any $k \times k$ matrix B, $\det B^T = \det B$.

Consider $A \in \mathbb{R}^{(k+1)\times(k+1)}$, i.e., for $n=k+1$.

Recall that Row i of A is the same as column i of A^T. Hence the cofactor matrix A_{ij} of A will be the transpose of the corresponding cofactor matrix of A^T. For instance, consider A_{i1}:



$$A^T_{i1} = B_{1i}$$

By induction assumption, the cofactors are same in the expansion for det A along Row i and det A^T along column i. Hence the result holds for $n = k+1$, and hence for all n.

Theorem 6 DL-LAA pg 196

$$\det AB = \det A \det B \quad \text{for } A, B \in \mathbb{R}^{n \times n}$$

Warning! $\det(A+B) \neq \det A + \det B$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

det: $1 \qquad 3 \quad \neq \quad 7$

What is $\det A^{-1}$? (Assuming $\det A \neq 0$)

$$\det A^{-1} = \frac{1}{\det A} \quad \text{if } \det A \neq 0.$$

As $AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det I = 1$

$\Rightarrow \det A \cdot \det A^{-1} = 1.$ $\Rightarrow \det A = \dfrac{1}{\det A^{-1}}$.

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Formula for $\det(rA)$ when $A \in \mathbb{R}^{n\times n}$.

multiply each entry by r.

$$\det(rA) = (r)^n \det A. \qquad A = [a_{ij}]$$

Expanding along Row 1,

$(n-1)\times(n-1)$ matrix

$$\det(rA) = \sum_{j=1}^{n} (-1)^{1+j}(ra_{1j})\cdot\det(rA_{1j})$$

For each dimension, we get a factor of r multiplying det A to get det rA. Hence,

$$\det(rA) = (r)^n \det A.$$

Prob 40 pg 200

$A, B \in \mathbb{R}^{4\times 4}$ with $\det A = -1$, $\det B = 2$.

(a) $\det AB = \det A \cdot \det B = -1\times 2 = -2$

(b) $\det B^5 = (\det B)^5 = 2^5 = 32$

(c) $\det B^{-1}AB = \cancel{\det B}\cdot\det A\cdot\dfrac{1}{\cancel{\det B}} = -1.$

Computer Project – Write your own rref function

Row reduction algorithm $\rightarrow$ a step-by-step procedure to row reduce any $m \times n$ matrix to its echelon form.

Iteration $\rightarrow$ repeat a step or calculation for several indices (or for several rows or columns).

e.g., Scale Row 1 of $A \in \mathbb{R}^{m \times n}$ by k.

With $A = [a_{ij}]$, we want $a_{11} \rightarrow k a_{11}$, $a_{12} \rightarrow k a_{12}, \dots,$

$a_{1n} \rightarrow k a_{1n}$.

"replace"

$$\left.\begin{array}{l} \textbf{for } j = 1 \text{ to } n \\ \quad a_{1j} \rightarrow k a_{1j} \\ \textbf{end} \end{array}\right\} \text{ a "for" loop}$$