MATH 230 – Lecture 21 (03/29/2011)

Today's office hours – 2:45-4:00 pm and then from 5:00-6:00 pm.

Vector Spaces (Section 4.1)

Def: A real vector space is a non-empty set of objects V, on which two operations addition and scalar multiplication are defined, and the objects satisfy the following ten axioms. Here, $\bar{u}, \bar{v}, \bar{w} \in V$, and $c, d \in \mathbb{R}$.

$\in$ element of

1. $\bar{u}+\bar{v} \in V$.
2. $\bar{u}+\bar{v} = \bar{v}+\bar{u}$.
3. $(\bar{u}+\bar{v})+\bar{w} = \bar{u}+(\bar{v}+\bar{w})$.
4. $\exists\ \bar{0} \in V$, the **zero** of V, such that $\bar{u}+\bar{0} = \bar{0}+\bar{u} = \bar{u}$.

   $\exists$ There exists

5. $\forall\ \bar{u} \in V$, $\exists$ an object $-\bar{u} \in V$ such that $\bar{u}+-\bar{u}=\bar{0}$

   $-\bar{u}$ is the **inverse** of $\bar{u}$.

6. $c\bar{u} \in V$.

7. $c(\bar{u}+\bar{v}) = c\bar{u} + c\bar{v}$.

8. $(c+d)\bar{u} = c\bar{u} + d\bar{u}$.

9. $c(d\bar{u}) = (cd)\bar{u}$

10. $1\bar{u} = \bar{u}$.

The main axioms are 1, 4, 6, i.e., V should have a zero, it should be closed under and under scalar multiplication.

e.g., $\mathbb{R}^2$, $\mathbb{R}^n$, $\{\bar{0}\}$ are vector spaces.

$\{\bar{0}\}$ → single element set

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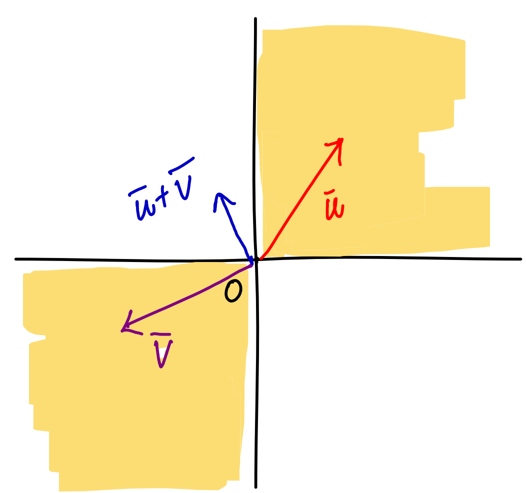
W is the union of first and third quadrants in $\mathbb{R}^2$.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, xy \geq 0 \right\}$$



(a) If $\bar{u} \in W$, $c \in \mathbb{R}$, is $c\bar{u} \in W$?

$\bar{u} \in W \Rightarrow \bar{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $xy \geq 0$

$c\bar{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}$. Hence the product $(cx)(cy) = c^2 xy \geq 0$ $\forall c \in \mathbb{R}$.

(b) But W is not closed under addition. Let

$\bar{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$. Both $\bar{u}, \bar{v} \in W$. But

$\bar{u} + \bar{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin W$.

Hence W is not a vector space.

For $n \geq 0$, $\mathbb{P}_n$ is the collection of all polynomials of degree up to n (including n).

$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ is an n^{th} degree polynomial if $a_n \neq 0$.

Note: $p(x) = a_0 + \cdots + a_n x^n$ is the same as $p(t)$.

Claim: $\mathbb{P}_n$ is a vector space.

Def: $p(t) = a_0 \neq 0$ is a zero-degree polynomial

$p(t) = 0$ is the **zero polynomial** $\in \mathbb{P}_n$.

But degree of the zero polynomial is undefined.

Let $q(t) = b_0 + b_1 t + \cdots + b_n t^n$

Then $p(t) + q(t)$ is the sum of the two polynomials.

$$(p+q)(t) = p(t) + q(t) = (a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_n t^n)$$

$$= (a_0 + b_0) + (a_1 + b_1) t + \cdots + (a_n + b_n) t$$

$$= c_0 + c_1 t + \cdots + c_n t^n \in \mathbb{P}_n.$$

Similarly, $c\,p(t)$ for $c \in \mathbb{R}$ is

$$c(a_0 + \cdots + a_n t^n) = c a_0 + \cdots + c a_n t^n \in \mathbb{P}_n.$$

Hence $\mathbb{P}_n$ is a vector space.