MATH 230 – Lecture 22 (03/31/2011)

$\mathbb{P}_n$ is a vector space (continued...)

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n, \quad a_j \in \mathbb{R} \text{ for } j = 0, \ldots, n$$

For any $c \in \mathbb{R}$, $c\,p(t) = c(a_0 + \cdots + a_n t^n)$

$$= ca_0 + \cdots + ca_n t^n$$

$$= c_0 + \cdots + c_n t^n = q(t)$$

Here $c_j = c\,a_j$; hence $q(t) \in \mathbb{P}_n$.

All other axioms are satisfied by polynomials.

For instance, $-p(t) = -(a_0 + \cdots + a_n t^n)$ is such that

$p(t) + -p(t) = 0$, the zero polynomial.

Prob 25, Page 224

Show that the zero of a vector space is unique.

Def. $\bar{0}$ is an element of V such that $\forall\, \bar{u} \in V$ "for all"

$$\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}.$$

Suppose there is a $\bar{w} \in V$, such that $\forall \bar{u} \in V$, $\bar{w}$ is different from $\bar{0}$

$$\bar{w} + \bar{u} = \bar{u} + \bar{w} = \bar{u}.$$

This result holds for $\bar{0} \in V$ as well. Hence

$$\bar{0} + \bar{w} = \bar{0} \implies \bar{w} = \bar{0},$$ i.e, the zero of V is unique (as $\bar{w}$ is not another zero, it's just identical to $\bar{0}$).

proof by contradiction. → $\bar{w} = \bar{0}$, which contradicts our starting assumption that they are different.

Subspaces ~ "a subset of V, which is a vector space"

Def. A subspace of a vector space V is a subset H of V ($H \subseteq V$) such that

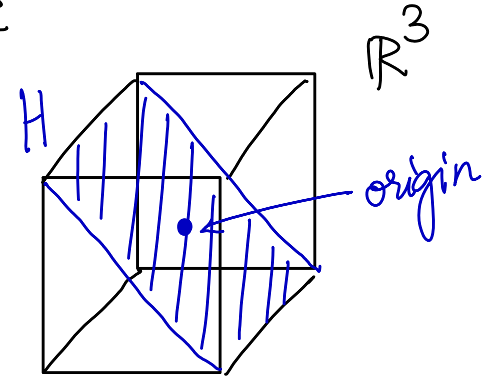
subset or equal to

(a) $\bar{0} \in H$, where $\bar{0}$ is the zero of V;

(b) $\forall\, \bar{u}, \bar{v} \in H$, $\bar{u} + \bar{v} \in H$, i.e., H is closed under addition; and

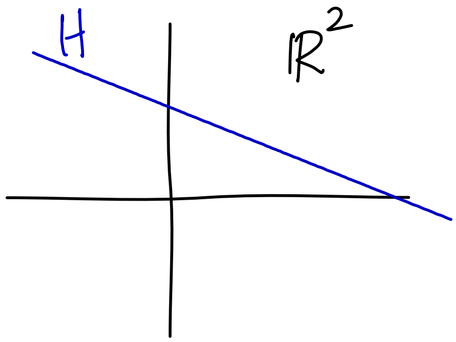
(c) $\forall\, \bar{u} \in H$, $c \in \mathbb{R}$, $c\bar{u} \in H$, i.e., H is closed under scalar multiplication.

Examples



H is a plane in $\mathbb{R}^3$ passing through the origin.

H is a subspace of $\mathbb{R}^3$.



H is a line not passing through the origin. Then H is not a subspace of $\mathbb{R}^2$.

Prob 13 pg 223

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \text{ and } \bar{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

(a) Is $\bar{w}$ in $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$? How many vectors are there in $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

No. $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is just the collection of the vectors $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$.

For any set S, $|S|$ denotes the # elements in S.

$|\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}| = 3.$

$\longrightarrow$ notation for size of a set, i.e, # entries in the set.

(b) Is $\bar{w}$ in $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$?

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array}\right] \xrightarrow{R_3-5R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has many solutions. Hence $\bar{w} \in \text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$.

(c) How many vectors are there in $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$?

$|\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)| = \infty$, as there are infinitely many linear combinations of $\bar{v}_1, \bar{v}_2, \bar{v}_3$.

<u>Prob 7, pg 223</u> Let W be the set of all polynomials of degree at most 3, with integer coefficients. Is W a subspace of $\mathbb{P}_n$ for $n \geq 3$?

collection of all polynomials with degree up to n.

$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, $a_j \in \mathbb{Z}$
$j = 0, 1, 2, 3$

set of all integers

W is **not closed under scalar multiplication**, as $ca_j \notin \mathbb{Z}$ when c is a fraction. $\bar{0} \in W$, and W is closed under addition. Hence W is not a subspace of $\mathbb{P}_n$, $n \geq 3$.

e.g., $p(t) = 1 + 2t \in W$, but $\frac{1}{3} p(t) = \frac{1}{3} + \frac{2}{3} t \notin W$.

Prob 8 pg 233

W is the set of all polynomials $p(t)$ in $\mathbb{P}_n$ such that $p(0) = 0$. Is W a subspace of $\mathbb{P}_n$?

$0 \in W$, as $p(0) = 0$ is present in W.

W closed under addition? $p(0) + q(0) = 0 + 0 = 0$.

YES.

W closed under scalar multiplication? $c\,p(0) = c \cdot 0 = 0$.

YES.

So, W is a subspace of $\mathbb{P}_n$.

need not be vectors

Theorem 1, DL-LAA pg 221 If $\underbrace{\bar{v}_1, \dots, \bar{v}_p}$ $\in V$, a vector space, then $H = \text{span}(\bar{v}_1, \dots, \bar{v}_p)$ is a subspace of V.

Proof $\bar{0} \in H$, as $\text{span}(\bar{v}_1, \dots, \bar{v}_p) = \left\{ \sum_{j=1}^{p} c_j \bar{v}_j \;\middle|\; c_j \in \mathbb{R} \right\}$.

Taking $c_j = 0$ for all j gives $\bar{0}$, the zero of V.

H is closed under addition. For $\bar{u}, \bar{w} \in H$,

$$\bar{u} = \sum_{j=1}^{p} c_j \bar{v}_j, \quad \bar{w} = \sum_{j=1}^{p} d_j \bar{v}_j, \text{ we get}$$

$$\bar{u} + \bar{w} = \sum_{j=1}^{p} (c_j + d_j) \bar{v}_j \in H$$

H is closed under scalar multiplication. For $\bar{u} \in H$, $d \in \mathbb{R}$ with $\bar{u} = \sum_{j=1}^{p} c_j \bar{v}_j$, $d\bar{u} = \sum_{j=1}^{p} (dc_j) \bar{v}_j \in H$.

<u>Prob 17, pg 223</u>

"such that"

$$W = \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$$ Is W a subspace of $\mathbb{R}^4$?

If yes, find a set of vectors S, such that $W = \text{span}(S)$.

$\bar{0} \in W$; take $a = b = c = 0$.

⋮

But, we could rather just find S such that $W = \text{span}(S)$.

This result would confirm that W is a subspace

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\bar{v}_1} + b\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_2} + c\underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_3} \quad ; \quad a, b, c \in \mathbb{R}$$

Hence $W = \text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$, where $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

Hence, W is a subspace of $\mathbb{R}^4$.

<u>Prob 16, pg 223</u>

$W = \left\{ \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}, \ a, b \in \mathbb{R} \right\}$. Is W a subspace of $\mathbb{R}^3$.

No, as $\bar{0} \notin W$. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$ has no solutions a, b.

$\left.\begin{aligned} a-6b &= 0 \\ 2b+a &= 0 \end{aligned}\right\}$ gives $a = b = 0$ as the unique solution, which does not agree with $-a+1=0$, which needs $a=1$.