# MATH 230 – Lecture 23 (04/05/2011)

Office hours today: 2:45–4:00 pm and 5–6:00 pm.

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Recall: H is a subspace of a vector space V, if

(i) $\bar{0} \in H$,

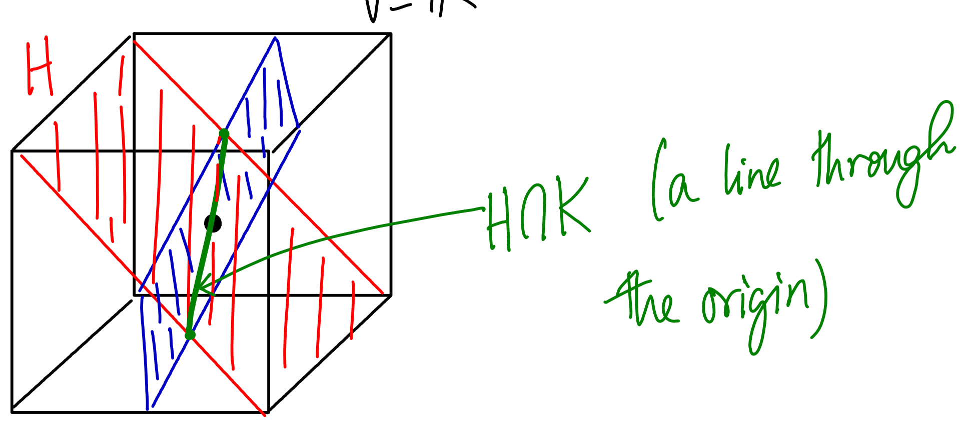
(ii) $\forall\, \bar{u}, \bar{v} \in H$, $\bar{u} + \bar{v} \in H$,

(iii) $\forall\, \bar{u} \in H$, $c \in \mathbb{R}$, $c\bar{u} \in H$.

## Prob 32 pg 225

Let H and K be subspaces of a vector space V. The set $H \cap K$ (intersect) is the collection of all elements of V that are in both H and K. Show that $H \cap K$ is a subspace of V. (intersection of subspaces is also a subspace).

Example: $V = \mathbb{R}^3$



Proof We show that $H \cap K$ satisfies the three axioms for being a subspace, i.e., it includes zero, and is closed under addition and scalar multiplication.

$H$ and $K$ are subspaces of $V$.

$$\Rightarrow \left.\begin{array}{l}\bar{0} \in H \\ \bar{0} \in K\end{array}\right\} \Rightarrow \bar{0} \in H \cap K.$$

Consider $\bar{u}, \bar{v} \in H \cap K$. Hence $\bar{u}, \bar{v} \in H$ and $\bar{u}, \bar{v} \in K$.

$$\left.\begin{array}{l}\forall\, \bar{u}, \bar{v} \in H,\ \bar{u}+\bar{v} \in H \\ \forall\, \bar{u}, \bar{v} \in K,\ \bar{u}+\bar{v} \in K\end{array}\right\} \Rightarrow \forall\, \bar{u}, \bar{v} \in H \cap K,\ \bar{u}+\bar{v} \in H \cap K.$$

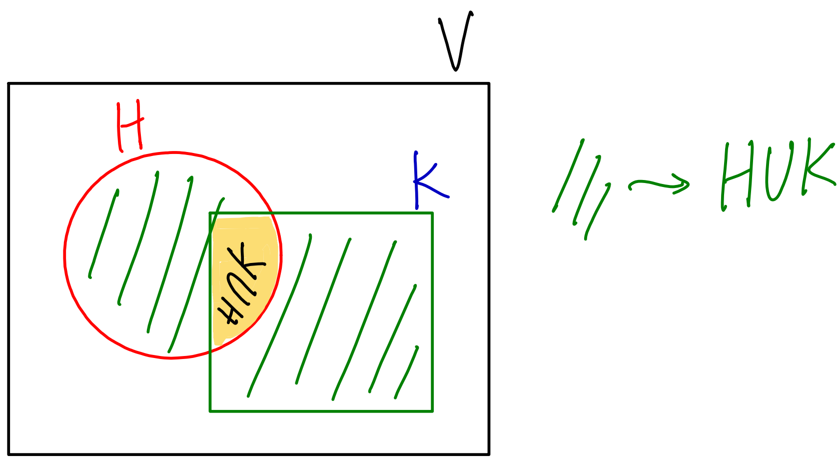
Consider $\bar{u} \in H \cap K$, $c \in \mathbb{R}$. So, $\bar{u} \in H$, and $\bar{u} \in K$.

$$\left.\begin{array}{l}\forall\, \bar{u} \in H,\ c \in \mathbb{R},\ c\bar{u} \in H \\ \forall\, \bar{u} \in K,\ c \in \mathbb{R},\ c\bar{u} \in K\end{array}\right\} \Rightarrow \forall\, \bar{u} \in H \cap K,\ c \in \mathbb{R},\ c\bar{u} \in H \cap K.$$

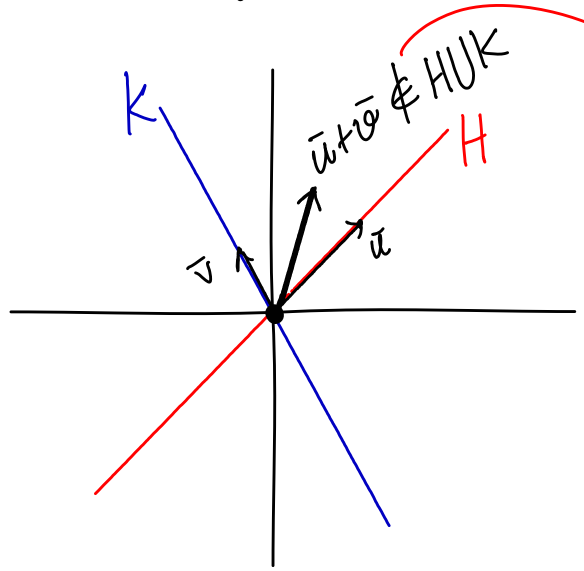
Hence $H \cap K$ is a subspace of $V$.

What about $H \cup K$?

$\cup$ "union" → collection of all entries in $V$ that are present in either $H$ or $K$, but not necessarily in both.




A Venn diagram illustrating union and intersection of sets.




$\notin$ "not element of" or "not in"

The union of two lines, which are both subspaces, is not closed under addition.

As illustrated by this example in $\mathbb{R}^2$, the union of two subspaces is typically not a subspace.

# Null Space and Column Space of A (Section 4.2)

$A \in \mathbb{R}^{m\times n}$

**Def** The **null space** of $A$, denoted by Nul $A$ or Nul$(A)$, is the set of all solutions to $A\bar{x}=\bar{0}$.

$$\text{Nul } A = \{\bar{x}\in\mathbb{R}^n \mid A\bar{x}=\bar{0}\}$$

**Theorem 2, DL-LAA pg 227** Nul $A$ is a subspace of $\mathbb{R}^n$.

**Proof** (i) $\bar{0}\in$ Nul $A$, as $A\bar{0}=\bar{0}$. ($\bar{0}$ is the trivial solution to $A\bar{x}=\bar{0}$.)

(ii) Let $\bar{u},\bar{v}\in$ Nul $A$. Then $A\bar{u}=0$, $A\bar{v}=\bar{0}$.

$\Rightarrow A\bar{u}+A\bar{v} = A(\bar{u}+\bar{v}) = \bar{0}$, i.e., $\bar{u}+\bar{v}\in$ Nul $A$.

(iii) Let $\bar{u}\in$ Nul $A$, $c\in\mathbb{R}$. Then $A\bar{u}=\bar{0}$.

$\Rightarrow cA\bar{u} = A(c\bar{u}) = \bar{0}$, i.e., $c\bar{u}\in$ Nul $A$.

Hence, Nul $A$ is a subspace of $\mathbb{R}^n$.

An explicit description of Nul $A$ is obtained by finding the parametric-vector form of all solutions to $A\bar{x}=\bar{0}$.

Prob 4, Pg 234 Find an explicit description of Nul A by listing a set of vectors that span Nul A, where

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \xrightarrow[\text{then } \frac{1}{2}R_2]{R_1 - 2R_2} \begin{array}{c} \quad x_2 \qquad x_4 \text{ free} \\ \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$x_1 = 6x_2$

$x_3 = 0$, $\quad x_2, x_4 \in \mathbb{R}$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4, \quad x_2, x_4 \in \mathbb{R}.$$

Hence $\text{Nul}\, A = \text{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, which is a subspace of $\mathbb{R}^4$.

Prob 8 Pg 234

$W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} \middle| \, 5r - 1 = s + 2t \right\}$. Is $W$ a subspace of $\mathbb{R}^3$?

$W$ is not a subspace, as $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$.

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not satisfy $5r - 1 = s + 2t$.

<u>Prob 9, Pg 234</u>

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; \begin{matrix} a-2b=4c \\ 2a = c+3d \end{matrix} \right\}.$$ Is $W$ a subspace of $\mathbb{R}^4$?

$$\left.\begin{matrix} a-2b=4c \\ 2a=c+3d \end{matrix}\right\} \Rightarrow \left.\begin{matrix} a-2b-4c \phantom{-3d} = 0 \\ 2a \phantom{-2b} - c - 3d = 0 \end{matrix}\right\} \Rightarrow$$

$$A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \bar{0} \quad \text{for } A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}.$$

Hence $W = \text{Nul}\, A$, and hence it is a subspace of $\mathbb{R}^4$.

<u>Prob 1 pg 234</u> $\quad \bar{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$. Is $\bar{w} \in \text{Nul}\, A$?

$$A\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ So $\bar{w} \in \text{Nul}\, A$.

As illustrated here, it is relatively easy to check if $\bar{w} \in \text{Nul}\, A$. We just calculate $A\bar{w}$.

## <u>Column Space of A</u>

The **column space** of $A$, denoted by $\text{Col}\, A$ or $\text{Col}(A)$, is the set of all linear combinations of the columns of $A$.

$$\mathrm{Col}\,A = \{\bar{b} \in \mathbb{R}^m \mid A\bar{x} = \bar{b} \text{ for some } \bar{x} \in \mathbb{R}^n\}$$

set of all right hand-sides for which $A\bar{x} = \bar{b}$ is consistent.

If $A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$, then

$$\mathrm{Col}\,A = \left\{ \sum_{j=1}^{n} \bar{a}_j x_j \;\middle|\; x_j \in \mathbb{R}\ \forall j \right\}$$

Also, Col A = range of T, where $T(\bar{x}) = A\bar{x}$, i.e., the set of all images of T.

$$\mathrm{Col}\,A = \mathrm{Span}(\bar{a}_1, \ldots, \bar{a}_n).$$

hence Col A is a subspace of $\mathbb{R}^m$!

<u>Prob 15, pg 234</u>

$$W = \left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} \;\middle|\; r,s,t \in \mathbb{R} \right\}.$$

Find A such that W = Col A.

$$\left.\begin{matrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{matrix}\right\} \overset{\text{"equivalent"}}{\equiv} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} r + \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} s + \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} t = A\bar{x}, \text{ where}$$

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \text{ and } \bar{x} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}.$$

Hence W = Col A.

Since $\text{Col}\,A = \text{span}(\bar{a}_1, \dots, \bar{a}_n)$, $\text{Col}\,A$ is a subspace of $\mathbb{R}^m$ (as each $\bar{a}_j \in \mathbb{R}^m$).

<u>Prob 23 pg 235</u> $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, $\bar{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Is $\bar{w} \in \text{Col}\,A$?

Solve $A\bar{x} = \bar{w}$

$$\left[\begin{array}{cc|c} -6 & 12 & 2 \\ -3 & 6 & 1 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & 6 & 1 \end{array}\right]$$

System is consistent. Hence $\bar{w} \in \text{Col}\,A$.

Unlike in the case of checking whether $\bar{w} \in \text{Nul}\,A$, to check if $\bar{w} \in \text{Col}\,A$, we typically have to do more work — solve $A\bar{x} = \bar{w}$.

But in the above example, one could notice that $A = [\bar{a}_1\ \bar{a}_2]$, where $\bar{a}_1 = -3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\bar{a}_2 = 6\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Hence, $\text{Col}\,A = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$, and as such, $\bar{w} \in \text{Col}\,A$.

We could not make such easy observations on most larger examples, though.