MATH 230 – Lecture 24 (04/07/2011)

Col A and Nul A of $A \in \mathbb{R}^{m\times n}$

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$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$ Is $\bar{w}$ in Col A? Is $\bar{w}$ in Nul A?

$\bar{w} \in \text{Col}A$ if $A\bar{x} = \bar{w}$ is consistent.

$$\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \xrightarrow[R_2 - \frac{3}{2}R_3]{R_1 + 2R_3} \left[\begin{array}{ccc|c} 0 & -2 & -1 & -2 \\ 0 & 4 & 2 & 4 \\ 4 & 0 & 4 & -2 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|c} 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & -2 \end{array}\right]$$

System is consistent. Hence $\bar{w} \in \text{Col}\,A$.

$$A\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$ so $\bar{w} \in \text{Nul}\,A$.

#from OL-LAA pg 232

Nul A v/s	Col A $\quad A \in \mathbb{R}^{m\times n}$
1. Subspace of $\mathbb{R}^n$	1. Subspace of $\mathbb{R}^m$
5. $\bar{w} \in \text{Nul}\,A$ if $A\bar{w} = \bar{0}$	2. $\bar{w} \in \text{Col}\,A$ if $A\bar{x} = \bar{w}$ is consistent
7. $\text{Nul}\,A = \{\bar{0}\}$ if there is a pivot in every column	7. $\text{Col}\,A = \mathbb{R}^m$ if A has a pivot in every row.

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$$x_1 - 3x_2 - 3x_3 = 0$$
$$-2x_1 + 4x_2 + 2x_3 = 0$$
$$-x_1 + 5x_2 + 7x_3 = 0$$

$\bar{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is a solution for the given system. Explain why $\bar{\mathbf{x}}' = \begin{bmatrix} 30 \\ 20 \\ -10 \end{bmatrix}$ is also a solution.

The system is $A\bar{x} = \bar{0}$, for $A = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 2 \\ -1 & 5 & 7 \end{bmatrix}$. $\bar{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ is a solution $\Longrightarrow$ $\bar{\mathbf{x}} \in \text{Nul } A$. Hence $c\bar{\mathbf{x}} \in \text{Nul } A$ for all $c \in \mathbb{R}$ (as Nul A is closed under scalar multiplication).

Nul A is a subspace

So $\bar{\mathbf{x}}' = 10\bar{\mathbf{x}}$ is also a solution to $A\bar{x} = \bar{0}$.

Linearly Independent Set and Bases (Section 4.3)

plural for Basis

Recall $\{\bar{v}_1, \ldots, \bar{v}_n\}$ with $\bar{v}_j \in \mathbb{R}^m$ is LI if $n \le m$ and $A\bar{x} = \bar{0}$ with $A = [\bar{v}_1 \cdots \bar{v}_n]$ has only the trivial solution (A has a pivot in every column).

Def Let H be a subspace of a vector space V. $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_p\}$ with $\bar{b}_j \in V$ for all j, is a **basis** of H if

(i) $\mathcal{B}$ is a linearly independent (LI) set, and

(ii) $\text{span}(\bar{b}_1, \dots, \bar{b}_p) = H$.

A set $\{\bar{v}_1, \dots, \bar{v}_p\}$ (not necessarily of vectors), is linearly dependent (LD) if $\bar{v}_j = \sum_{i \neq j} c_i \bar{v}_i$ for some j, i.e., one object is a linear combination of the other objects.

$c_i \in \mathbb{R}$

A set that is not LD is linearly independent (LI).

e.g., colors $\{\overset{R}{\text{Red}}, \overset{G}{\text{Green}}, \overset{B}{\text{Blue}}\}$ is LI, but $\{R, G, B, \text{Purple}\}$ is LD as we can combine Red and Blue to get Purple.

Examples of Bases

① $\{\bar{e}_1, \dots, \bar{e}_n\}$ (the set of unit vectors) is a basis for $\mathbb{R}^n$.

→ unit vectors

→ The standard basis of $\mathbb{R}^n$.

② $\{1, t, t^2, \dots, t^n\}$ is a basis for $\mathbb{P}_n$.

$$p(t) = a_0 + a_1 t + \dots + a_n t^n = a_0(1) + a_1(t) + \dots + a_n(t^n),$$

$a_0, a_1, \dots, a_n \in \mathbb{R}$.

→ standard basis for $\mathbb{P}_n$

③ $\{1, t, 2t^2\}$ is a basis for $\mathbb{P}_2$.

$\{-2, 4t+6, \sqrt{2}t^2\}$ is also a basis for $\mathbb{P}_2$.

Note: This set is indeed LI, as we cannot write any one element as a linear combination of the others — the degrees do not match.

Also, every polynomial $p(t)$ with degree ≤ 2 can be written as $p(t) = a_0(-2) + a_1(4t+6) + a_2(\sqrt{2}t^2)$ for $a_0, a_1, a_2 \in \mathbb{R}$.

e.g., if $p(t) = 2+t$, we need $a_2 = 0$, $4a_1 = 1$, and $6a_1 - 2a_0 = 2$, which give $a_0 = -\frac{1}{4}$, $a_1 = \frac{1}{4}$, $a_2 = 0$.

Bases for Col A and Nul A

Theorem 6 DL-LAA pg 241 The pivot columns of A form a basis for Col A.

the columns in original matrix A, not in its echelon form.

Check: (i) Pivot columns are LI.

(ii) Col A = span(pivot columns)

as non-pivot columns can be written as linear combinations of pivot columns.

Basis for Nul A $\rightsquigarrow$ from parametric vector form of solutions to $A\bar{x} = \bar{0}$.

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pivot columns

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad B \sim A.$$

row equivalent.

Find bases for Col A and Nul A.

Since columns 1 and 2 are pivot columns,

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$$ is a basis for Col A.

Need rref(A) for basis of Nul A.

$$B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \times 1/2} \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ x_3, x_4 free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} x_4,$$ $x_3, x_4 \in \mathbb{R}$ are all solutions to $A\bar{x} = \bar{0}$

Hence $$\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for Nul A.

Def The number of elements in a basis for subspace H is called its **dimension**.

Result Each basis of H has the same number of elements.

e.g., 1. $\mathbb{R}^2$ is 2-dimensional. (dimension of $\mathbb{R}^2$ is 2).

e.g., $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$ are bases.

2. $\mathbb{P}_2$ is 3-dimensional.

e.g., $\underbrace{\{1, t, t^2\}}_{\text{3 elements.}}$ is a basis.