MATH230 – Lecture 26 (04/14/2011)

Dimension of a vector space = # elements in any basis

Prob 19, pg 244

$\bar{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, $H = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.

It can be shown that $4\bar{v}_1 + 5\bar{v}_2 - 3\bar{v}_3 = \bar{0}$. Find a basis for H (without doing EROs).

$\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LD, so cannot be a basis for H.

We can see that $\bar{v}_i \neq c\bar{v}_j$ for $i,j = 1,2,3$, $i \neq j$.

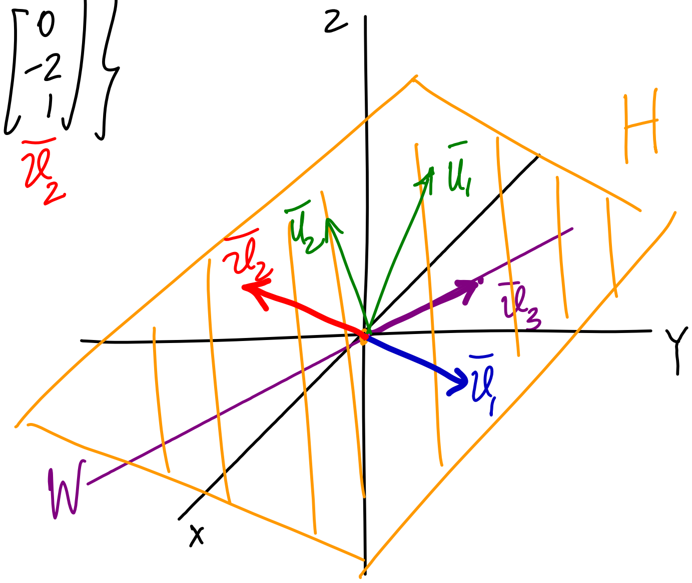
Hence $\{\bar{v}_1, \bar{v}_2\}$ is LI, and $\text{span}\{\bar{v}_1, \bar{v}_2\} = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ (since $4\bar{v}_1 + 5\bar{v}_2 - 3\bar{v}_3 = \bar{0}$). Hence $\{v_1, v_2\}$ span H, and hence is a basis for H.

Similarly, we can pick $\{\bar{v}_1, \bar{v}_3\}$ or $\{\bar{v}_2, \bar{v}_3\}$ as bases for H.

Here, $\dim H$ = dimension of $H = 2$.

$H = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

$\dim H = 2$ as $\left\{ \underset{\bar{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}, \underset{\bar{v}_2}{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}} \right\}$ is a basis.



$W = \text{span}\left\{ \underset{\bar{v}_3}{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}} \right\}$ $\dim W = 1.$

$\text{Span}\{\bar{u}_1, \bar{u}_2\} = H$

<u>Prob 12, pg 261</u>

$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$

Find the dimension of the subspace spanned by these vectors. $\to H$

Note: $\dim H < 4$, as 4 vectors in $\mathbb{R}^3$ are LD.

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 + 2R_3} \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 0 & 0 & 8 \\ 0 & 1 & 5 & 7 \end{bmatrix}$$

Hence $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}$ is a basis for H. $\dim H = 3.$

If $H = \{\bar{0}\}$, $\dim H = 0$, as $\{\bar{0}\}$ is not LI.

Basis Theorem (Theorem 12, DL-LAA pg 259)

Let V be a vector space with $\dim V = p \geq 1$. Any LI subset of exactly p elements in V is a basis for V. Also, any subset of p elements that span V is a basis for V.

* $\dim V = p \rightsquigarrow$ # elements in a basis
* basis has LI elements
* basis spans V.

$\to$ two of these three results imply the third.

If $A \in \mathbb{R}^{m \times n}$

$\dim \operatorname{Col} A$ = # pivot columns in $A \leq m$

$\dim \operatorname{Nul} A$ = # free variables in $A \leq n$

(we can have all n variables free if A is the $m \times n$ zero matrix).

Prob 21, pg 261

Show that $\{1, 2t, -2+4t^2, -12t+8t^3\}$ forms a basis for $\mathbb{P}_3$.

Hermite polynomials

Note: $\dim \mathbb{P}_3 = 4$. In general, $\dim \mathbb{P}_n = n+1$.

"corresponds to"

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \Longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Hence $\{1, 2t, -2+4t^2, -12t+8t^3\} \Longleftrightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix} \right\}$.

$$\left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_j \in \mathbb{R}, j = 0,1,2,3 \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix} \right\},$$

○ : pivots

and these four vectors are LI.

Hence $\{1, 2t, -2+4t^2, -12t+8t^3\}$ span $\mathbb{P}_3$.

(prob 15, pg 181, section 2.9)

$A \in \mathbb{R}^{3\times 5}$, has 3 pivot columns.

Is $\operatorname{Col} A = \mathbb{R}^3$? Is $\operatorname{Nul} A = \mathbb{R}^2$? What are $\dim \operatorname{Col} A$ and $\dim \operatorname{Nul} A$?

$\operatorname{Col} A = \mathbb{R}^3$. The columns of A are in $\mathbb{R}^3$ as A is 3×5. Since A has 3 pivots, there is a pivot in every row, hence the columns span $\mathbb{R}^3$, i.e., $\operatorname{Col} A = \mathbb{R}^3$.

$\operatorname{Nul} A$ is a subspace of $\mathbb{R}^5$, and hence cannot be $\mathbb{R}^2$. But $\dim \operatorname{Nul} A = 2$ here.

as there are 2 free variables.

$\dim \operatorname{Col} A = \#$ pivot columns $= 3$.

(Section 4.6)

Def $A \in \mathbb{R}^{m\times n}$, $\dim \operatorname{Col} A$ is called the **rank** of the matrix A (written as $\operatorname{rank} A$ or $\operatorname{rank}(A)$).

So, $\operatorname{rank} A = \#$ pivot columns in A.

Rank Theorem (Theorem 14, DL-LAA pg 265)

If $A \in \mathbb{R}^{m \times n}$, then

$$\boxed{\text{rank } A + \dim \text{Nul } A = n}$$

Prob Create a 3×4 matrix A of rank 2. What is $\dim \text{Nul } A$?

$$A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ has 2 pivots and hence $\text{rank } A = 2$

Since $\text{rank } A + \dim \text{Nul } A = n = 4$,

$\dim \text{Nul } A = 2$.

The columns 1, 3, 5, and 6 of A are LI, and $\text{rank } A = 4$. Explain why these four columns must be a basis for $\text{Col } A$.

Since $\text{rank } A = \dim \text{Col } A = 4$, any 4 LI columns of A form a basis for $\text{Col } A$ (Basis theorem).

Invertible Matrix Theorem (IMT)

See Lecture 17 for original statement of IMT.

(a) $A \in \mathbb{R}^{n\times n}$ is invertible

(m) Columns of A form a basis for $\mathbb{R}^n$.

(n) $\operatorname{Col} A = \mathbb{R}^n$

(o) $\dim \operatorname{Col} A = n$

(p) $\operatorname{rank} A = n$

(q) $\operatorname{Nul} A = \{\bar{0}\}$

(r) $\dim \operatorname{Nul} A = 0$.