MATH230 - Lecture 27 (04/19/2011)

Eigenvalues and Eigenvectors (Chapter 5)

$A \in \mathbb{R}^{n \times n}$ LT $\bar{x} \longmapsto A\bar{x}$ We know how to find a basis for the range of this LT ($=$ Col A).

What can we say about vectors $\bar{x}$ for which $T(\bar{x})$ is just a scalar multiple of $\bar{x}$?

$$A\bar{x} = \lambda \bar{x} \text{ for some } \lambda \in \mathbb{R}.$$

Def An n-vector $\bar{x}$ is an **eigenvector** of A if it is non-zero, and for some scalar λ, we have $A\bar{x} = \lambda\bar{x}$. λ is called an **eigenvalue** of A, and $\bar{x}$ is an eigenvector corresponding to the eigenvalue λ.

Note: λ can be zero. $\bar{x}$ has to be nonzero.

$\bar{x} = \bar{0}$ always satisfies $A\bar{x} = \lambda\bar{x}$. Hence we only look for <u>non-zero</u> $\bar{x}$ (for eigenvectors).

non-trivial solutions to some homogeneous system.

Some 2x2 examples

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ The matrix of the LT $T: \mathbb{R}^2 \to \mathbb{R}^2$ which reflects on the $45°$ line $y = x$.

Find $\bar{x} \neq \bar{0}$ and λ such that $A\bar{x} = \lambda \bar{x}$. There are two unknowns: $\bar{x}$ and λ.

$A\bar{x} - \lambda\bar{x} = \bar{0}$, i.e., $A\bar{x} - \lambda I \bar{x}$ where I is the 2×2 identity matrix.

$\Rightarrow (A - \lambda I)\bar{x} = \bar{0}$ To get $\bar{x} \neq \bar{0}$ as solutions, we need $A - \lambda I$ not invertible, i.e., $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}. \Rightarrow \det(A - \lambda I) = \lambda^2 - 1$$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$ There are two eigenvalues, $\lambda = +1$, $\lambda = -1$.

For $\lambda = 1$, we can find an eigenvector by solving

$(A - \lambda I)\bar{x} = \bar{0}$ i.e, $(A - I)\bar{x} = \bar{0} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

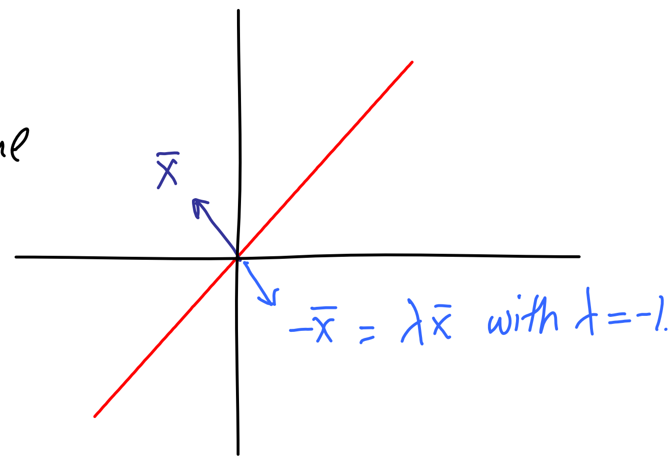
$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow[R_1 \rightleftarrows R_2]{R_1 + R_2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ x_2 free, $x_1 = x_2$ $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$, $x_2 \in \mathbb{R}$

So, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

$\lambda = -1$: $(A - \lambda I)\bar{x} = \bar{0} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ x_2 free. $x_1 = -x_2$ $\bar{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$, $x_2 \in \mathbb{R}$.

So, $\bar{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -1$.



(b) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$

3+3

$(3-\lambda)^2 - 1 = 0$

$\lambda^2 - 6\lambda + \underbrace{(9-1)}_{\det A = 3 \times 3 - 1 \times 1} = 0 \rightarrow (\lambda - 2)(\lambda - 4) = 0$

$\lambda = 2, 4$ are eigenvalues.

In general, for $A \in \mathbb{R}^{2 \times 2}$, the equation for λ looks like $\lambda^2 - \underbrace{(\text{sum of entries in diagonal of } A)}_{\text{trace}(A)}\lambda + \det A = 0$

trace$(A) \rightarrow$ sum of diagonal entries of an $n \times n$ matrix A.

(c) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation about origin by 90° CCW.

$$\det(A-\lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0 \qquad \lambda^2 + 1 = 0$$

No real eigenvalues

Result: If A is symmetric, i.e., $A^T = A$, or $A_{ij} = A_{ji}$ for all i,j, then A has only real eigenvalues.

If A is antisymmetric, i.e., $A^T = -A$, then all eigenvalues of A are imaginary.

Note: $A_{ij} = -A_{ji}$ if A is antisymmetric. Hence when $i=j$, $A_{ii} = -A_{ii} \implies A_{ii} = 0$. Hence antisymmetric matrices have zeroes on the diagonal.

We will worry only about symmetric matrices, and hence with real eigenvalues, in Math 230.

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Is $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ an eigen vector of $A=\begin{bmatrix}3&6&7\\3&3&7\\5&6&5\end{bmatrix}$? If yes, find the corresponding eigenvalue.

$$\bar{u}=\begin{bmatrix}1\\-2\\1\end{bmatrix}. \quad A\bar{u}=\begin{bmatrix}3&6&7\\3&3&7\\5&6&5\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}-2\\4\\-2\end{bmatrix}=-2\begin{bmatrix}1\\-2\\1\end{bmatrix}=-2\bar{u}.$$

So, $\bar{u}$ is an eigenvector of A with $\lambda=-2$ being the eigenvalue.

Prob 8 pg 308 Is $\lambda=3$ an eigenvalue of $A=\begin{bmatrix}1&2&2\\3&-2&1\\0&1&1\end{bmatrix}$? If yes, find a corresponding eigenvector.

Check if $\det(A-\lambda I)$ is zero.

$$A-3I=\begin{bmatrix}-2&2&2\\3&-5&1\\0&1&-2\end{bmatrix} \quad \det(A-\lambda I)=(-1)^{3+2}\cdot 1\cdot\begin{vmatrix}-2&2\\3&1\end{vmatrix}$$

$$+(-1)^{3+3}\cdot(-2)\cdot\begin{vmatrix}-2&2\\3&-5\end{vmatrix}$$

$$=8-8=0. \text{ Yes!}$$

To find an eigenvector, find a non-trivial solution to $(A-\lambda I)\bar{x}=\bar{0}$

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 \times -\frac{1}{2}} \begin{bmatrix} 1 & -1 & -1 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 + 2R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_2 \rightleftarrows R_3]{R_1 + R_3} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 free, $\bar{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} x_3$, $x_3 \in \mathbb{R}$.

So, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 3$.

Def: **Eigenspace** of A corresponding to eigenvalue λ is the set of all eigenvectors corresponding to the eigenvalue λ plus the **zero vector** (eventhough zero vector is not an eigen vector).

Prob 10, pg 308 Find a basis for the eigenspace of $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 4$.

Find a basis for $\text{Nul}(A - \lambda I)$.

$$A - \lambda I = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} \xrightarrow{R_2 - \frac{2}{3}R_1} \begin{bmatrix} 6 & -9 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \times \frac{1}{6}} \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$$

$\bar{x} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} x_2$, $x_2 \in \mathbb{R}$. Hence $\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$ is a basis for the corresponding eigenspace.

dimension of eigenspace is 1.

Eigenvalues of triangular matrices

Recall: If A is a triangular matrix, $\det A$ is just the product of entries on the diagonal.

$$A = \begin{bmatrix} a & * & * & \cdots & * \\ 0 & b & & & \vdots \\ 0 & & c & & \vdots \\ \vdots & & & \ddots & * \\ 0 & 0 & 0 & & m \end{bmatrix}$$ (upper triangular)

$$\det A = a \cdot b \cdot c \cdots m$$

What about $\det(A-\lambda I)$?

$$A-\lambda I = \begin{bmatrix} a-\lambda & & * \\ & b-\lambda & \\ & & \ddots & \\ 0 & & & m-\lambda \end{bmatrix},$$ hence

$$\det(A-\lambda I) = (a-\lambda)(b-\lambda)\cdots(m-\lambda).$$

Hence $a, b, c, \dots, m$ are roots of the equation

$\det(A-\lambda I) = 0$. So,

Eigenvalues of triangular matrices are the entries on the diagonal.

Def: $\det(A-\lambda I)$ is called the **characteristic polynomial** of A, and $\det(A-\lambda I)=0$ is the **characteristic equation** of A. (In MATLAB, look for charpoly(·)). eig(·) gives both eigenvalues and eigenvectors).

Prob 10 pg 317 Find the characteristic polynomial of

$$A=\begin{bmatrix}0&3&1\\3&0&2\\1&2&0\end{bmatrix}.$$

Note: A is symmetric here.

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda&3&1\\3&-\lambda&2\\1&2&-\lambda\end{vmatrix}=(-\lambda)\begin{vmatrix}-\lambda&2\\2&-\lambda\end{vmatrix}-3\begin{vmatrix}3&2\\1&-\lambda\end{vmatrix}$$

$$+1\begin{vmatrix}3&-\lambda\\1&2\end{vmatrix}$$

$$=(-\lambda)(\lambda^2-4)-3(-3\lambda-2)+(6+\lambda)$$

$$\vdots$$