MATH230 — Lecture 28 (04/21/2011)

Characteristic polynomial of $A \in \mathbb{R}^{n\times n}$ is $\det(A-\lambda I)$

Characteristic equation : $\det(A-\lambda I)=0$.

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Note: A is symmetric here.

$$A=\begin{bmatrix}0&3&1\\3&0&2\\1&2&0\end{bmatrix}.$$

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda&3&1\\3&-\lambda&2\\1&2&-\lambda\end{vmatrix}=(-1)^{2+1}\cdot 3\cdot\begin{vmatrix}3&1\\2&-\lambda\end{vmatrix}+(-1)^{2+2}(-\lambda)\cdot\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix}$$

$$+(-1)^{2+3}(2)\cdot\begin{vmatrix}-\lambda&3\\1&2\end{vmatrix}$$

$$=-3(-3\lambda-2)-\lambda(\lambda^2-1)-2(-2\lambda-3)$$

$$=-\lambda^3+(9+1+4)\lambda+(6+6)=-\lambda^3+14\lambda+12$$

Eigenspace corresponding to eigenvalue λ

def 1: Vector space spanned by all linearly independent eigenvectors corresponding to eigenvalue λ.

def 2: Collection of all eigenvectors plus the zero vector.

even though $\vec{0}$ is not an eigenvector

def 3: Nullspace of $(A-\lambda I)$.

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$A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Without calculation, find an eigenvalue and two linearly independent eigenvectors (corresponding to that eigenvalue) of A.

Notice that $A\bar{x} = \bar{0}$ will have non-trivial solutions. We can have only one pivot, and hence get two free variables.

Hence $\lambda = 0$ is an eigenvalue, because with $\lambda = 0$, $A\bar{x} = \lambda\bar{x}$ is the same as $A\bar{x} = \bar{0}$. Hence all non-trivial solutions to $A\bar{x} = \bar{0}$ are eigenvectors of A corresponding to $\lambda = 0$.

We can choose $\bar{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\bar{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ as two LI eigenvecters.

Another eigenvalue is $\lambda = 15$, as $\bar{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ gives

$A\bar{x}_3 = A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15\bar{x}_3$. So, $\bar{x}_3$ is an eigenvector corresponding to eigenvalue $\lambda = 15$.

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$$A=\begin{bmatrix}5&0&0&0\\8&-4&0&0\\0&7&1&0\\1&-5&2&1\end{bmatrix}$$

List all eigenvalues along with their multiplicities.

Since A is triangular, the eigenvalues are on the diagonal. They are $5(1)$, $-4(1)$, and $1(2)$, with the multiplicities listed in $(\cdot)$.

Def: The **multiplicity** of an eigenvalue is the # times it appears as a root of the characteristic equation.

e.g., $(\lambda-3)^2(\lambda+2)(\lambda+4)^3=0 \longrightarrow$ characteristic equation

The eigenvalues and their multiplicities are $3(2)$, $-2(1)$, and $-4(3)$.

Prob 26 Page 309 Show: If A^2 is the zero matrix, then the only eigenvalue of A is 0.

λ is an eigenvalue of $A \Longrightarrow A(\underbrace{A\bar{x}=\lambda\bar{x}})$ — $\bar{x}\neq\bar{0}$, as it's an eigenvector

$$\Rightarrow \underbrace{A^2}_{\bar{0}}\bar{x}=\lambda\underbrace{A\bar{x}}_{\lambda\bar{x}} \quad \Longrightarrow \bar{0}=\lambda\lambda\bar{x}=\lambda^2\bar{x}$$

$\Rightarrow \lambda^2\bar{x}=\bar{0}$ with $\bar{x}\neq\bar{0}$. Hence we must have $\lambda^2=0$, i.e., $\lambda=0$.

Similarity

Def $A, B \in \mathbb{R}^{n\times n}$. A is similar to B if there is an invertible matrix P such that

$$B = P^{-1}AP \quad \text{and} \quad A = PBP^{-1}.$$

Result If A is similar to B, then A and B have the same characteristic polynomial and hence the same set of eigenvalues.

Proof $B = P^{-1}AP$ for invertible P.

Want to show $\det(A-\lambda I) = \det(B-\lambda I)$

$$B-\lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda \underbrace{P^{-1}P}_{I}$$

$$= P^{-1}(AP-\lambda P) = P^{-1}(A-\lambda I)P$$

So, $B-\lambda I$ is similar to $A-\lambda I$.

Taking determinants, we get

$$\det(B-\lambda I) = \det\left(P^{-1}(A-\lambda I)P\right)$$

$$= \det(P^{-1})\det(A-\lambda I)\cdot\det(P)$$ as $\det(AB) = \det A \cdot \det B$

$$= \frac{1}{\cancel{\det(P)}}\cdot\det(A-\lambda I)\cdot\cancel{\det(P)}$$ as P is invertible $\det(P) \neq 0$

$$\Rightarrow \det(B-\lambda I) = \det(A-\lambda I).$$

The QR algorithm to estimate the eigenvalues of A

Prob 23 pg 318 If $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.

Note that both Q and R are $n\times n$ when $A \in \mathbb{R}^{n\times n}$

Want to show: There is invertible matrix P such that

$$A_1 = P^{-1}AP$$

$$A_1 = RQ = \underbrace{Q^{-1}Q}_{I\text{, as } Q \text{ is invertible}}RQ = Q^{-1}(QR)Q = Q^{-1}AQ$$

So A_1 and A are similar.

<u>The idea of QR algorithm</u>: Factorize A into Q_1R_1 where $Q_1^T = Q_1^{-1}$ and R_1 is upper triangular. So, $A = Q_1R_1$. Write $A_1 = R_1Q_1$, and repeat, i.e., write $A_1 = Q_2R_2$ with $Q_2^T = Q_2^{-1}$, R_2 upper triangular. Then set $A_2 = R_2Q_2$, and so on.

After sufficient # iterations (k), the diagonal entries of R_k will be close to the eigenvalues of A. Higher the k, closer the diagonal entries of R_k are to eigenvalues of A. Hence the use of the word "estimate".