MATH230 — Lecture 29 (04/26/2011)

QR algorithm $(A \in \mathbb{R}^{n \times n})$

$A = Q_1 R_1$ where $Q_1^T = Q_1^{-1}$ and R_1 is upper triangalur.

set $A_1 = R_1 Q_1$

Decompose $A_1 \to Q_2 R_2$, write $A_2 = R_2 Q_2$, and so on.

Diagonal entries of A_k approximate eigenvalues of A.

In MATLAB, use the function qr to find the Q and R.

(MATLAB demo → see course web page)

Prob 18 Pg 318

It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in matrix A below for which the eigenspace corresponding to $\lambda = 5$ is 2-dimensional.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $\lambda = 5$, $A - \lambda I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 0 & 0 & 6-h & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \rightleftarrows R_1} \begin{bmatrix} 0 & -2 & h & 0 \\ 0 & 0 & 6-h & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We need $h = 6$ so that there are two free variables, and hence the dimension of the eigenspace is 2.

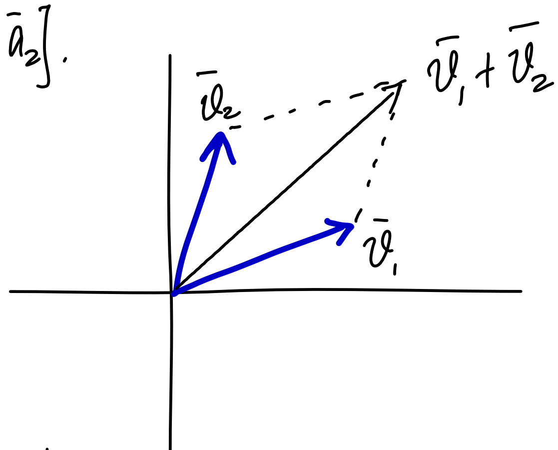
(Notice that since A is 4×4, we need two pivots so that $\dim \operatorname{Nul} A = n - 2 = 4 - 2$ is $= 2$).

We could have made the same conclusion by looking at the columns of A. We need two pivots. The fourth column has one pivot. Between columns 2 & 3, we need one pivot, which is possible when $h = 6$.

Prob 22, pg 318 (True/False)

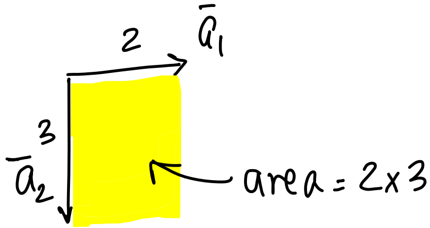
(a) If $A \in \mathbb{R}^{3\times 3}$, then $\det A$ is the volume of the parallelepiped determined by $\bar{a}_1, \bar{a}_2, \bar{a}_3$ where $A = [\bar{a}_1\ \bar{a}_2\ \bar{a}_3]$.

In 2D, two LI vectors determine a parallelogram.



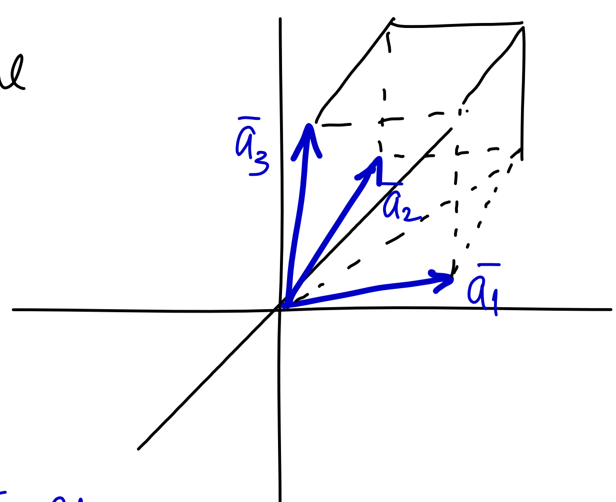
Another example: $\bar{a}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, $A = [\bar{a}_1\ \bar{a}_2]$.

$\det A = \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -6$



So $|\det A|$ gives the area of the rectangle.

In general, $|\det A|$ gives the area of the parallelogram formed by $\bar{a}_1$ and $\bar{a}_2$. In fact, in arbitrary dimensions, $|\det A|$ gives the volume of the parallelepiped formed by $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$.



As given, the statement is FALSE, as $|\det A|$, and not $\det A$, gives the volume.

When $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\det A = 0$. Indeed, the parallelogram generated by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is just a squished line segment. Hence its area is zero.

(d) A row replacement ERO on A does not change its eigenvalues.

FALSE. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $\det(A - \lambda I) = (1-\lambda)(1-\lambda) = \lambda^2 - 2\lambda + 1$

$$A \xrightarrow{R_1 + R_2} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = A' \quad \det(A' - \lambda I) = (2-\lambda)(1-\lambda) - 1 \times 1$$
$$= \lambda^2 - 3\lambda + 2 - 1 = \lambda^2 - 3\lambda + 1$$

So, eigenvalues of A and A' are not same.

But we could have examples where replacement EROs do not change the eigenvalues.

e.g., $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A - \lambda I) = (1-\lambda)^2$

$$A \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A'. \quad \det(A' - \lambda I) = (1-\lambda)^2$$

Theorem 2 (Section 5.1)

If $\bar{v}_1, \dots, \bar{v}_r$ are eigenvectors of a matrix A corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then $\{\bar{v}_1, \dots, \bar{v}_r\}$ is LI.

Proof for 2 vectors: Let $\bar{v}_1, \bar{v}_2$ be eigenvectors corresponding to λ_1, λ_2 with $\lambda_1 \neq \lambda_2$. Then we need to show $\{\bar{v}_1, \bar{v}_2\}$ is LI, i.e., $\bar{v}_1 \neq c\bar{v}_2$ for $c \in \mathbb{R}$.

Assume $\{\bar{v}_1, \bar{v}_2\}$ is LD, i.e., $\bar{v}_1 = c\bar{v}_2$.

$$A(\bar{v}_1 = c\bar{v}_2)$$

$\Rightarrow A\bar{v}_1 = cA\bar{v}_2$ But $A\bar{v}_1 = \lambda_1\bar{v}_1$ and $A\bar{v}_2 = \lambda_2\bar{v}_2$.

$\Rightarrow \lambda_1\bar{v}_1 = c\lambda_2\bar{v}_2$

Case 1: $\lambda_1 = 0$ or $\lambda_2 = 0$; Say $\lambda_1 = 0$. Then we get

$0 = c\lambda_2\bar{v}_2$ i.e., $\bar{v}_2 = \bar{0}$.