MATH 230 - Lecture 2 (01/13/2011)

Elementary row operations (EROs) can be applied to ANY matrix. We do so on augmented matrices of systems of linear equations to find their solution(s).

Prob 14, pg 11

Solve the given system.

$$\begin{aligned} x_1 - 3x_2 \quad &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \rightleftarrows R_3} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

$$\xrightarrow[R_3+2R_2]{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow{R_3 \times \left(\frac{1}{7}\right)} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-3R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow$$

This is the augmented matrix of a simpler system:

$$\begin{aligned} x_1 \quad &= 2 \\ x_2 \quad &= -1 \\ x_3 &= 1 \end{aligned}$$

Note that this system belongs to Case 2, i.e., it has a unique solution.

Two matrices are **row equivalent** if we get one from the other using EROs. If we get the second matrix from the first using EROs, we can get the first from the second using another set of EROs.

EROs are reversible.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{R_1+3R_3}{\overset{R_1-3R_3}{\rightleftarrows}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Two systems of linear equations are **equivalent** if they have the same solution set. Note that in this case, their augmented matrices are row equivalent.

<u>Prob 11, pg 11</u>

Solve the system:

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{array}\right] \xrightarrow{R_1 \rightleftarrows R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{array}\right] \xrightarrow{R_3-3R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

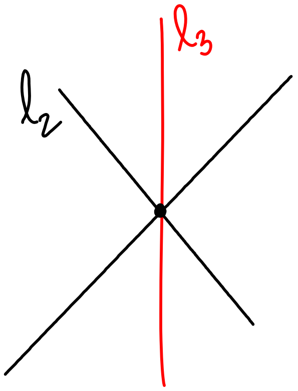
This row corresponds to $0x_1 + 0x_2 + 0x_3 = 2$, i.e, $0 = 2$!

Hence the original system is inconsistent, i.e., it has no solution.

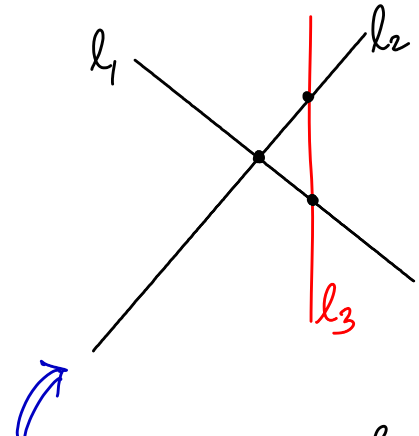
In general, if you get **[0 0 ... 0|*]** for *** non-zero**, the original system is **inconsistent.**

What if we get [0 0 ... 0 | 0] ?

This is o.k., i.e., the system is consistent, as we just have $0=0$, which is true. Such a row means that one equation in the original system is "redundant".



This is the graph of of 3 equations in two variables. The system is consistent, as ALL 3 lines meet at the same point. We get this point by using lines (ℓ_1, ℓ_2), (ℓ_1, ℓ_3), or (ℓ_2, ℓ_3) alone. Hence, one equation is redundant.



This is the graph of a system of 3 equations in 2 variables that is inconsistent. Notice that all three lines do not meet at the same point. But every pair of lines gives a consistent system.

Existence and uniqueness questions

1. Is the system consistent?
2. If there is a solution, is it the unique solution?

We'll see these questions again and again and again...

Prob 20, pg 12

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$ For what values of h is this the augmented matrix of a consistent system?

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$$

can be any value!

We cannot get $[0\ 0 | *]$ for $*$ nonzero. Hence the system is consistent for all values of h, ie, $h \in \mathbb{R}$

"in" or "element of".

set of all real numbers

Prob 25, pg 12

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

Find a relation between g, h, k such that this is the augmented matrix of a consistent system.

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$$

We need $k+2g+h=0$ for the system to be consistent. Else, we get $[0\ 0\ 0 | *]$ for $*$ nonzero.

TRUE/FALSE (Prob 23, pg 12)

Write justifications as illustrated here.

(b) A 5x6 matrix has 6 rows.

FALSE. It has 5 rows and 6 columns.

(*) A system of 3 linear equations in 3 variables can have 3 solutions.

FALSE. If it has more than one solution, it has infinitely many solutions, not 3.

Echelon form of a matrix (Section 1.2)

Def: **Non-zero row**: a row that has at least one nonzero entry

leading entry: The leftmost non-zero entry in a non-zero row.

A matrix is in **row echelon form**, or simply, in **echelon form**, if

(1) all nonzero rows are above any zero rows; and

(2) the leading entry of each row is in a column that is to the right of the leading entry of the row above it.

(1) & (2) imply that

(3) all entries in a column below a leading entry are zero.

A matrix is in **reduced row echelon form (RREF)** if it is in echelon form, and

(4) the leading entry in each non-zero row is 1, and

(5) the leading 1 is the only nonzero entry in its column. (all other entries are zero).

e.g., $\begin{bmatrix} 3 & 5 & 0 & 9 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ is in echelon form → "step like"

$\begin{bmatrix} 1 & 0 & \cancel{0}1 & 0 & 2 \\ 0 & 1 & \cancel{0}5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form.

still in reduced echelon form.

General notation for echelon forms

■ → non-zero, $*$ → zero or non-zero

$\begin{bmatrix} ■ & * & * \\ 0 & ■ & * \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form

$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form.

What are **all** possible echelon forms of a 3x3 matrix?

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any matrix can be converted to an echelon form, and to reduced echelon form using EROs.

A matrix can have several echelon forms, but its **reduced echelon form is unique.**

To solve a system of linear equations, we take its augmented matrix to echelon form, and to reduced echelon form if needed (i.e., if it is consistent).