MATH230 — Lecture 30 (04/28/2011)

Eigenvectors corresponding to distinct eigenvalues are LI.

$\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p, \bar{v}_{p+1}, \ldots$ are eigenvectors of A corresponding to distinct eigenvalues $\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}, \ldots}_{\lambda_i \neq \lambda_j \ \forall i,j}$.

Case of $\bar{v}_1, \bar{v}_2$

Assume $\{\bar{v}_1, \bar{v}_2\}$ is LD. Then $\bar{v}_1 = c\bar{v}_2$ for $c \in \mathbb{R}$.

$c \neq 0$ as both $\bar{v}_1$ and $\bar{v}_2$ are non-zero vectors

$A\left(\bar{v}_1 = c\bar{v}_2\right)$

$\Rightarrow A\bar{v}_1 = cA\bar{v}_2$ but $A\bar{v}_1 = \lambda_1\bar{v}_1$, $A\bar{v}_2 = \lambda_2\bar{v}_2$

$\Rightarrow \lambda_1\bar{v}_1 = c\lambda_2\bar{v}_2$

Case 1 One of λ_1, λ_2 is zero, say $\lambda_1 = 0$.

We get $\bar{0} = c\lambda_2\bar{v}_2 \Rightarrow \bar{v}_2 = \bar{0}$, but $\bar{v}_2 \neq \bar{0}$ as it is an eigenvector. We get a contradiction.

Case 2 $\lambda_1 \neq 0$, $\lambda_2 \neq 0$

$\Rightarrow \bar{v}_1 = \left(c\frac{\lambda_2}{\lambda_1}\right)\bar{v}_2 = c'\bar{v}_2$ where $c' = c\frac{\lambda_2}{\lambda_1} \neq c$ as $\lambda_1 \neq \lambda_2$

But $\bar{v}_1 = c\bar{v}_2$, hence we must have $\bar{v}_1 = \bar{v}_2 = \bar{0}$, giving again a contradiction.

Note: We must have $\lambda_1 \neq \lambda_2$ as they are distinct.

Proof in general

Let $\{\bar{v}_1, \dots, \bar{v}_p\}$ be LI, but $\bar{v}_{p+1}$ be a linear combination of $\bar{v}_1, \dots, \bar{v}_p$ for some $p \geq 2$.

$$\Rightarrow A\left(\bar{v}_{p+1} = c_1 \bar{v}_1 + \dots + c_p \bar{v}_p\right) \quad \text{———} \ (1)$$

$$\Rightarrow A\bar{v}_{p+1} = c_1 A\bar{v}_1 + \dots + c_p A\bar{v}_p$$

$$\Rightarrow \lambda_{p+1} \bar{v}_{p+1} = c_1 \lambda_1 \bar{v}_1 + \dots + c_p \lambda_p \bar{v}_p \quad \text{———} \ (2)$$

as $A\bar{v}_j = \lambda_j \bar{v}_j \ \forall j$

(2) $- \lambda_{p+1}$ (1) gives

$$\bar{0} = c_1 \underbrace{(\lambda_1 - \lambda_{p+1})}_{\neq 0} \bar{v}_1 + \dots + c_p \underbrace{(\lambda_p - \lambda_{p+1})}_{\neq 0} \bar{v}_p$$

as λ_j's are distinct

Since $\{\bar{v}_1, \dots, \bar{v}_p\}$ is LI, we must have $c_1 = \dots = c_p = 0$ here. But then $\bar{v}_{p+1} = c_1 \bar{v}_1 + \dots + c_p \bar{v}_p = \bar{0}$, which cannot be true, as $\bar{v}_{p+1}$ is an eigen vector. So we get a contradiction.

Review for Final (Problems from Practice Final).

Prob 3

Rank A = # pivot columns. We need a 3x3 matrix with 2 pivots here.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ has rank 2, as two columns are LI, but column 3 is the same as column 2.

take first column of A

$\bar{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \text{Nul } A$, as $A\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \bar{0}$.

Another choice is $\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, as $A\bar{b} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \neq \bar{0}$.

add all columns of A

Prob 8

$$\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}}_{\bar{v}_1} a + \underbrace{\begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}}_{\bar{v}_2} b + \underbrace{\begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}}_{\bar{v}_3} c$$

So, the set W of all vectors of the given form can be written as

$W = \text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$.

Notice that $\bar{v}_1 - 2\bar{v}_2 = \bar{v}_3$.

If you do not notice this relation, just check by the usual method if $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LI or not.

$$\begin{bmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ -1 & -4 & 7 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3+R_1 \\ R_4-3R_1}]{R_2-2R_1} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \\ 0 & -6 & 12 \\ 0 & 7 & -14 \end{bmatrix} \xrightarrow[R_4-(7/9)R_2]{R_3+\frac{6}{9}R_2} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But $\bar{v}_i \neq c\bar{v}_j$ for $i,j=1,2,3$, $i\neq j$ here. Hence $\{\bar{v}_1, \bar{v}_2\}$, $\{\bar{v}_1, \bar{v}_3\}$, $\{\bar{v}_2, \bar{v}_3\}$ are all bases for W.

<u>Prob 2</u> A has to be 2×2.

Given: $A\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$, and $A\begin{bmatrix}2\\7\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$.

$$\Rightarrow \quad A\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}=\begin{bmatrix}1 & 3\\1 & 1\end{bmatrix} \qquad \det\left(\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}\right)=1\neq 0$$

$$\text{Hence } \left(\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}\right)^{-1}=\frac{1}{1}\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}=\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}.$$

$$\text{Hence } A\underbrace{\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}}_{I}=\begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}$$

$$\Rightarrow \quad A=\begin{bmatrix}-2 & 1\\4 & -1\end{bmatrix}.$$

Prob 6

$p_1(t) = 1+t$, $p_2(t) = 1-t$, $p_3(t) = -4$, $p_4(t) = t+t^2$, $p_5(t) = 1+2t-2t^2$.

$\mathbb{P}_n$ is the set of all polynomials with degree upto n.

$$p(t) = a_0 + a_1 t + \dots + a_n t^n \equiv \bar{v} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

go to a vector corresponding to each polynomial

Here, we get

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_5 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}.$$

$$A = [\bar{v}_3 \ \bar{v}_1 \ \bar{v}_2 \ \bar{v}_4 \ \bar{v}_5] = \begin{array}{c} \begin{bmatrix} -4 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} p_3 & p_1 & p_2 & p_4 & p_5 \end{array} \end{array}$$

Hence $\{p_1(t), p_3(t), p_4(t)\}$ is a basis for H. Similarly, $\{p_2(t), p_3(t), p_4(t)\}$ is also a basis.

Note: We could take any three LI vectors out of the five, and pick the corresponding polynomials for a basis here.

We could have avoided the 4^{th} entry — which is uniformly zero here — and reached the same conclusions.

Prob 12, True/False

(a) FALSE. $A=\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$. $\Rightarrow$ rref $(A)=$ rref$(B)=B$.

But Col A is span$\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, while Col B $=$ span$\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

(b) TRUE. All columns of A are pivot columns. Hence there are no free variables. So $A\bar{x}=\bar{0}$ has only the trivial solution.

(c) TRUE. $(AB)^{-1}=B^{-1}A^{-1}$ Product of two matrices is invertible iff both are invertible.

Since AB^{-1} is invertible, so are A and B^{-1}.

Hence both A and B are invertible. So $(AB)^{-1}=B^{-1}A^{-1}$.

(d) FALSE. The result holds only if the plane passes through the origin.

(e) TRUE. $A^{-1}(A\bar{x}=\lambda\bar{x})$ $\qquad A^{-1}$ exists

$\bar{x}=\lambda A^{-1}\bar{x}$. Since $\lambda\neq 0$, we get $A^{-1}\bar{x}=\left(\frac{1}{\lambda}\right)\bar{x}$

So $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. $\bar{x}$ is an eigenvector of A and A^{-1} here.