MATH 230 – Lecture 3 (01/18/2011)

Echelon form and reduced echelon form

Row reduction is the procedure of using EROs to reduce **any** matrix into echelon form, and then to reduced echelon form.

Def :

pivot → the leading entry in a nonzero row

pivot position → position of a leading entry (same as pivot)

pivot column → a column that contains a pivot.

Prob 4, pg 25 Row reduce the given matrix to echelon form, and then to reduced echelon form. Circle the pivots, list the pivot columns.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

is in echelon form

$$\xrightarrow[R_3\times\left(-\frac{1}{10}\right)]{R_2\times\left(-\frac{1}{4}\right)} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-3R_3]{R_1+2R_3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot columns

is in reduced echelon form

Columns 1, 2, and 4 are pivot columns.

Solution of linear systems: Use row reduction to convert the augmented matrix to echelon form, and if found consistent, reduce further to reduced echelon form to write the solutions down.

<u>Prob 12, pg 25</u> The augmented matrix of a system of linear equations is given. Find all solutions.

pivots are circled

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{rhs} \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right] \xrightarrow{R_3+4R_2} \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot columns 1 & 3 correspond to variables x_1 and x_3.

Def. Variables corresponding to pivot columns are called **basic** variables. Variables that are not basic are **free** variables (or non-basic variables).

Here, x_1, x_3 are basic and x_2, x_4 are free variables.

We can choose the values of free variables arbitrarily. The basic variables can be written in terms of the free variables.

A system of linear equations has **infinitely many solutions** if and only if it **has free variables**

Back to prob 12, pg 25

The reduced echelon form is $\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

$$\left.\begin{aligned} x_1 - 7x_2 + 6x_4 &= 5 \\ x_3 - 2x_4 &= -3 \end{aligned}\right\} \quad \boxed{\begin{aligned} x_1 &= 5 + 7x_2 - 6x_4 \\ x_3 &= -3 + 2x_4 \\ &x_2, x_4 \text{ are free} \end{aligned}}$$

general solution to the original system.

Summary of linear systems

1. If the echelon form of the augmented matrix has a row $[0\ 0 \dots 0 \mid *]$ for $*$ non-zero, the system is inconsistent.

Else

2. If there are no free variables, the system has a unique solution.

3. If there are free variables, the system has infinitely many solutions.

Prob 15(a) pg 25

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$$

is the augmented matrix in echelon form. Is the system consistent?

YES, as there is no row of the form $[0\ 0\ 0\ *]$ for $*$ nonzero!

Is the solution unique?

YES. As there are no free variables.

Prob 25, pg 26 The coefficient matrix has a pivot in each row. Why is the system consistent?

There is at least one non-zero entry in each row of the coefficient matrix. Hence we cannot have an inconsistent row $\left([0\ 0 \dots 0 \mid \blacksquare] \right)$ in the augmented matrix.

Does this system have a unique solution?

Don't know.

There could be free variables even when there is a pivot in each row, in which case, the solution is not unique.

Prob 19, pg 26

$x_1 + hx_2 = 2$
$4x_1 + 8x_2 = k$

Choose h, k such that system is
(a) inconsistent;
(b) has unique solution; and
(c) has many solutions.

ALSO, in cases (b) and (c), write all solutions down.

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right] \xrightarrow{R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right]$$

(a) We need $8-4h=0$ and $k-8 \neq 0$, so that we get $[0\ 0 \mid \blacksquare]$. So, $h=2$ and $k \neq 8$.

can also write as $k \in \mathbb{R}/\{8\}$

set of all reals without 8.

(b) $8-4h \neq 0$ so that there are no free variables.
$h \neq 2$, k is any real number. $h \in \mathbb{R}/\{2\}$, $k \in \mathbb{R}$.

(c) Need $8-4h=0$ and $k-8=0$ so that x_2 is a free variable in a consistent system.
i.e., $h=2$, $k=8$.

Now, let's find the solution(s) in (b) and (c).

(b) We have $8-4h \neq 0$.

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right] \xrightarrow{R_2 \times \frac{1}{(8-4h)}} \left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 1 & \frac{k-8}{8-4h} \end{array}\right] \xrightarrow{R_1 - hR_2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2-h\left(\frac{k-8}{8-4h}\right) \\ 0 & 1 & \frac{k-8}{8-4h} \end{array}\right]$$

$$x_1 = 2 - h\left(\frac{k-8}{8-4h}\right) = \frac{2(8-\cancel{4h}) - hk + \cancel{8}h}{4(2-h)}$$

$$= \frac{16-hk}{4(2-h)}$$

$$x_2 = \frac{k-8}{8-4h} = \frac{k-8}{4(2-h)}$$

The unique solution is $x_1 = \frac{16-hk}{4(2-h)}$, $x_2 = \frac{k-8}{4(2-h)}$.

(c) $h=2$, $k=8$ gives

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free, x_1 is basic

$$\left.\begin{array}{l} x_1 + 2x_2 = 2 \\ x_2 \text{ free} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = 2 - 2x_2 \\ x_2 \text{ free} \end{array}\right\}$$

general solution.

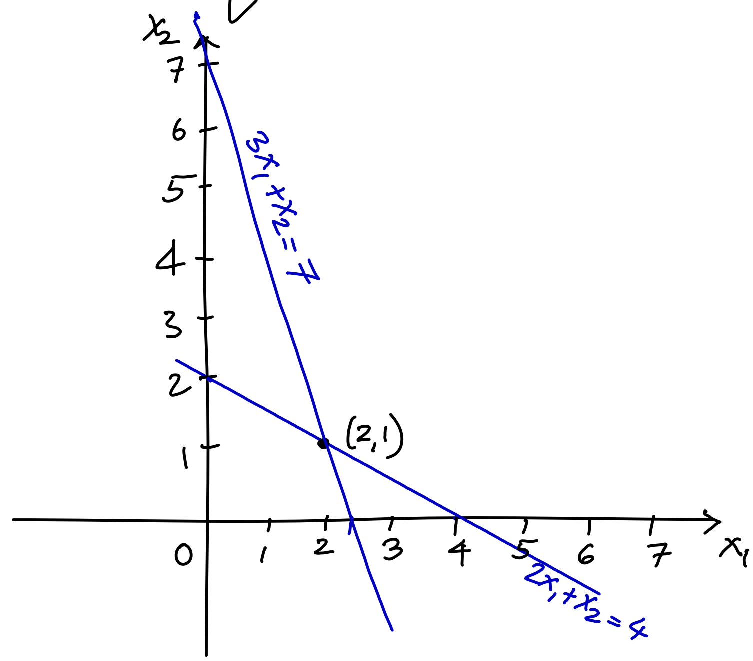
Vector Equations (Section 1.3)

$$\begin{aligned} 3x_1 + x_2 &= 7 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

$$\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 0 & -5 & -5 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[R_1 \rightleftarrows R_2]{R_1 \times \left(-\frac{1}{5}\right)}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad \begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases} \text{unique solution.}$$

This is the "row picture" of the system. We'll now look at the "column picture".



$\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right]$ corresponds to

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \Big\} \text{ is a vector equation}$$

vectors.

A vector (or a column vector, by default) is a column of entries. An n-vector is a column with n entries.

"bar": indicates a vector

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$\bar{x}$ here is also a $1 \times n$ matrix.