MATH 230 – Lecture 4 (01/20/2011)

Vector equations

$$\left.\begin{aligned} 3x_1 + x_2 &= 7 \\ x_1 + 2x_2 &= 4 \end{aligned}\right\} \qquad \begin{bmatrix}3\\1\end{bmatrix}x_1 + \begin{bmatrix}1\\2\end{bmatrix}x_2 = \begin{bmatrix}7\\4\end{bmatrix}$$

$x_1 = 2, x_2 = 1$ is the unique solution

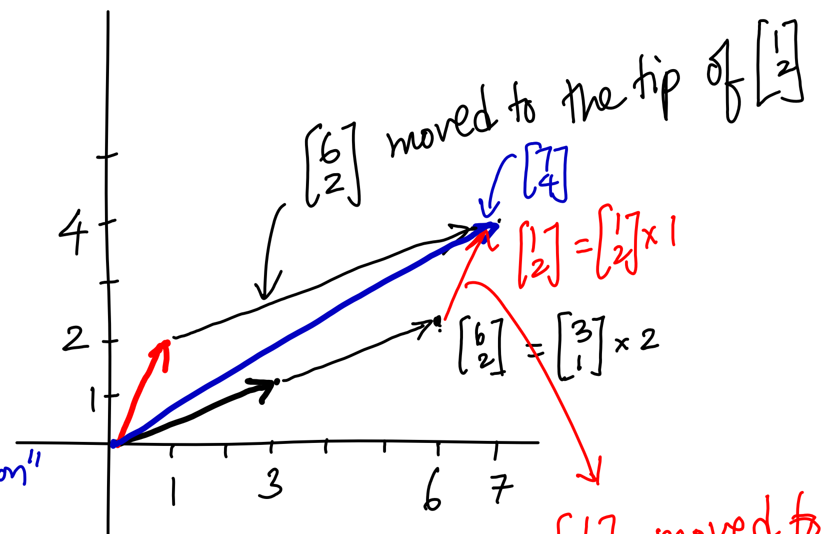
Notation: The set of all n-vectors is denoted by $\mathbb{R}^n$.

e.g., $\bar{x} = \begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \in \mathbb{R}^3$ when $\alpha, \beta, \gamma \in \mathbb{R}$.

$\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}7\\4\end{bmatrix}$ are vectors in $\mathbb{R}^2$.

$c\bar{x}$ for $c \in \mathbb{R}$, $\bar{x} \in \mathbb{R}^n$ is vector-scalar multiplication.

also called "scalar multiplication"



If $\bar{x} = \begin{bmatrix}x_1\\x_2\\ \vdots\\ x_n\end{bmatrix}$, then

$$c\bar{x} = \begin{bmatrix}cx_1\\cx_2\\ \vdots\\ cx_n\end{bmatrix}.$$

Also, $c\bar{x}$, "stretches" the vector $\bar{x}$ by the factor c.

In words, can you find multipliers x_1 and x_2, which when used to scale the vectors $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$, and added, gives $\begin{bmatrix}7\\4\end{bmatrix}$?

Note: $\bar{u} = [2\ 3\ 5]$ is a row-vector. It is **NOT** the same as $\bar{v} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, which is a column vector.

By default, we assume vectors are column vectors.

Adding scalar multiples of vectors is called "taking a linear combination" of the vectors.

Def: For vectors $\bar{v}_1, \bar{v}_2, \dots \bar{v}_n \in \mathbb{R}^m$, the vector

$$\bar{v} = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_n\bar{v}_n \text{ for } c_i \in \mathbb{R} \text{ for all } i, \text{ is}$$

a **linear combination** of $\bar{v}_1, \dots, \bar{v}_n$.

Note: We can add vectors $\bar{u}$ and $\bar{v}$ only if they have the same # entries.

So, $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ is not defined.

The c_i's in the linear combination $\bar{v} = c_1\bar{v}_1 + \dots + c_n\bar{v}_n$ are called **weights** of the linear combination.

e.g.,
$$-3\begin{bmatrix}3\\1\end{bmatrix}+4\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-3\times 3\\-3\times 1\end{bmatrix}+\begin{bmatrix}4\times 1\\4\times 2\end{bmatrix}=\begin{bmatrix}-9+4\\-3+8\end{bmatrix}=\begin{bmatrix}-5\\5\end{bmatrix}$$

So, $\begin{bmatrix}-5\\5\end{bmatrix}$ is a linear combination of $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ with weights $c_1=-3$ and $c_2=4$.

Note that $\left\{\begin{array}{l}3x_1+x_2=-5\\x_1+2x_2=5\end{array}\right\}$ has the solution $x_1=-3,\ x_2=4$.

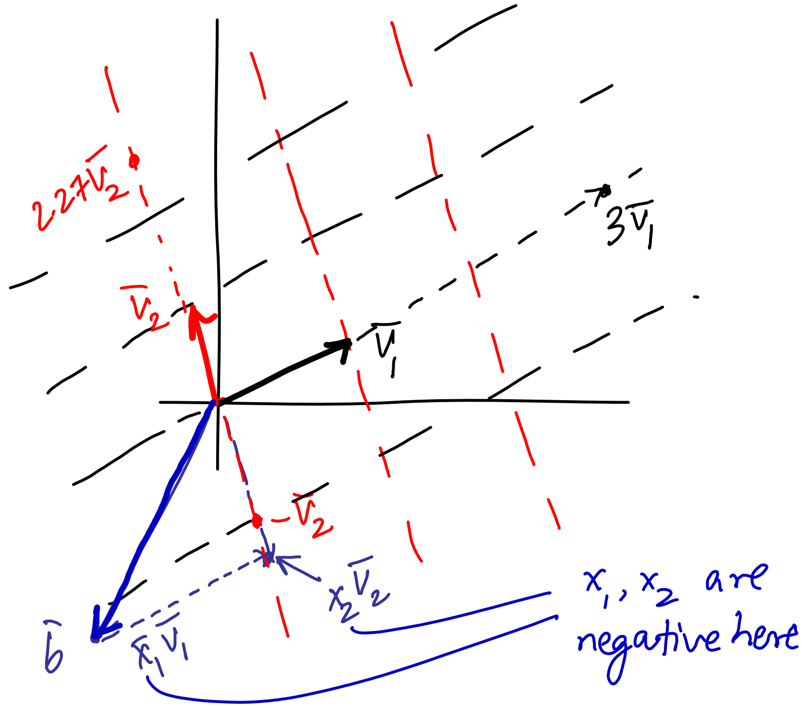
In general, if $\bar{b}=c_1\bar{a}_1+c_2\bar{a}_2+\cdots+c_n\bar{a}_n$, then the system of linear equations whose augmented matrix is $[\bar{a}_1\ \bar{a}_2\cdots\bar{a}_n\,|\,\bar{b}]$ has a solution $x_1=c_1, x_2=c_2, \dots, x_n=c_n$.

Equivalently, to answer the question "Is the system with augmented matrix $[\bar{a}_1\ \bar{a}_2\cdots\bar{a}_n\,|\,\bar{b}]$ consistent?", we can answer the question "**Can $\bar{b}$ be written as a linear combination of $\bar{a}_1,\bar{a}_2,\dots,\bar{a}_n$?**".

Def: The set of all linear combinations of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \in \mathbb{R}^m$ is called the **span** of $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$, and written $\text{span}\{\bar{v}_1, \ldots, \bar{v}_n\}$.

We can write any vector in $\mathbb{R}^2$ as a linear combination of $\begin{bmatrix}3\\1\end{bmatrix}$ & $\begin{bmatrix}1\\2\end{bmatrix}$.

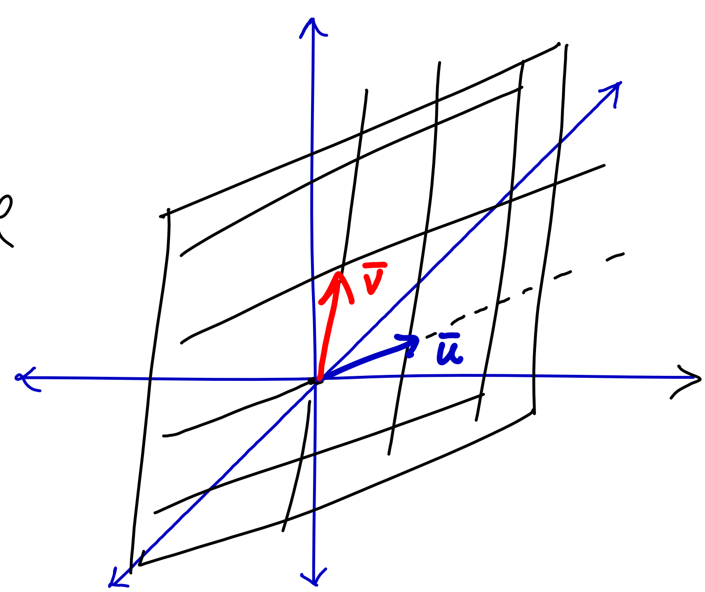
$$\text{span}\left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\} = \mathbb{R}^2.$$



Note that the zero vector, or origin, is in every span.

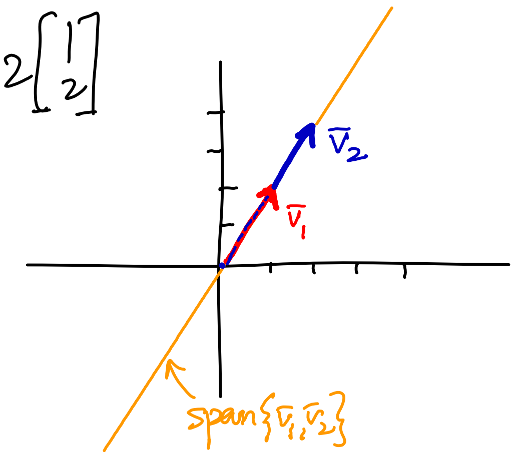
$$\bar{0} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} = 0\cdot\bar{v}_1 + 0\cdot\bar{v}_2 + \cdots + 0\cdot\bar{v}_n$$ (take all weights as zero).

In $\mathbb{R}^3$, the span of the two vectors $\bar{u}, \bar{v}$ shown here is a <u>plane</u> passing through the origin.



But $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$ is just a line in $\mathbb{R}^2$ through the origin, as

$$\bar{v} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\4\end{bmatrix} = \underbrace{c_3}_{[c_1+2c_2]}\begin{bmatrix}1\\2\end{bmatrix}, \text{ since } \begin{bmatrix}2\\4\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix}$$



Prob 18, pg 38

$\bar{v}_1 = \begin{bmatrix}1\\0\\-2\end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix}-3\\1\\8\end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix}h\\-5\\-3\end{bmatrix}$. For what values of h is $\bar{v}_3$ in the plane generated by $\bar{v}_1$ and $\bar{v}_2$?

refers to span$\{\bar{v}_1, \bar{v}_2\}$.

Reword: For what h is $\bar{v}_3$ in span$\{\bar{v}_1, \bar{v}_2\}$?

Equivalently for what h is $\left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\-2 & 8 & -3\end{array}\right]$ the augmented matrix of a consistent system?

$$\left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\-2 & 8 & -3\end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\0 & 2 & -3+2h\end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cc|c}1 & -3 & h\\0 & 1 & -5\\0 & 0 & 7+2h\end{array}\right]$$

Need $7+2h=0$, to avoid $[0 \; 0 \,|\, \blacksquare]$ row.

So, $h = \frac{-7}{2}$.

Properties of $\mathbb{R}^n$ (n-vectors)

$\bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^n$, $\underbrace{c, d}_{\text{scalars}} \in \mathbb{R}$, $\bar{0}$ is the zero vector

(i) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$

(ii) $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$

(iii) $\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}$

(iv) $\bar{u} + (-\bar{u}) = \bar{0}$

(v) $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$

(vi) $(c+d)\bar{u} = c\bar{u} + d\bar{u}$

(vii) $c(d\bar{u}) = (cd)\bar{u} = (c\bar{u})d$

These are properties of addition and multiplication for scalars (i.e., real numbers) extended to the case of vectors.

TRUE/FALSE (Prob 23, pg 38)

(c) An example of linear combination of $\bar{v}_1$ and $\bar{v}_2$ is $\frac{1}{2}\bar{v}_1$.

TRUE. $\frac{1}{2}\bar{v}_1 = \left(\frac{1}{2}\right)\bar{v}_1 + 0\,\bar{v}_2$ is a linear combination of $\bar{v}_1$ and $\bar{v}_2$ with weights $\frac{1}{2}, 0$.

(e) span$\{\bar{u}, \bar{v}\}$ is always a plane through origin.

FALSE. If $\bar{u}$ and $\bar{v}$ are scalar multipoles of each other, we get a line, not a plane — see page 4-5 above.

Introduction to MATLAB on http://my.math.wsu.edu.

rref(A) gives the reduced row echelon form of matrix A.