MATH 230 - Lecture 5 (01/25/2011)

The matrix equation $A\bar{x}=\bar{b}$ (Section 1.4)

For the system

$$\left\{\begin{array}{l} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{array}\right\}, \quad \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right] \longrightarrow \text{augmented matrix, and}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \longrightarrow \text{vector equation}$$

The matrix equation is $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

In general, the matrix equation is $A\bar{x}=\bar{b}$, where A is an $m\times n$ matrix with $A=[\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$ corresponding to the system whose augmented matrix is $[A\,|\,\bar{b}]$.

of m equations in n variables

$A\bar{x}$ is a **matrix-vector product**. In general, if $A=[\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$, where $\bar{a}_j \in \mathbb{R}^m$, and $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

then $$A\bar{x} = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n}_{\text{linear combination of the } \bar{a}_j\text{'s with weights } x_j\text{'s}}.$$

This is the linear combination of $\bar{a}_1, \dots, \bar{a}_n$ with weights $x_1, x_2, \dots, x_n$. Thus we get the result that $A\bar{x} = \bar{b}$ has a solution if and only if $\bar{b}$ can be written as a linear combination of $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$. Equivalently, if and only if $\bar{b} \in \text{span}\{\bar{a}_1, \dots, \bar{a}_n\}$.

Why study just A ?

Motivation: We can answer several questions about the system $A\bar{x} = \bar{b}$ just by doing EROs on A.

Note: The matrix-vector product $A\bar{x}$ is defined only when A is $m \times n$ and $\bar{x}$ is an n-vector, i.e., when

$$(\#\text{ columns in } A) = (\#\text{ entries in } \bar{x}).$$

e.g., $\cancel{\begin{bmatrix} -4 & 2 \\ 0 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}}$ is not defined.

$\begin{bmatrix} -4 & 2 \\ 0 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is defined, and is given by

$$\begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} \times 3 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \times (-2) = \begin{bmatrix} -4 \times 3 + 2 \times -2 \\ 0 \times 3 + 1 \times -2 \\ 6 \times 3 + 3 \times -2 \end{bmatrix} = \begin{bmatrix} -16 \\ -2 \\ 12 \end{bmatrix}$$

Prob 14, pg 48 Let $\bar{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$.

Is $\bar{u}$ in the subset of $\mathbb{R}^3$ spanned by columns of A?

Reword: Is $\bar{u}$ in span$\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$, where $A = [\bar{a}_1\ \bar{a}_2\ \bar{a}_3]$?

Or, does the system $A\bar{x} = \bar{u}$ have a solution?

$$\left[\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{array}\right] \xrightarrow{R_1 \rightleftarrows R_3} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{bmatrix} \xrightarrow{R_3 - 5R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{bmatrix}$$

$$\xrightarrow{R_3 + 7R_2} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{bmatrix}$$

the system is inconsistent.

So, $\bar{u} \notin \text{span}\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$ for $A = [\bar{a}_1\ \bar{a}_2\ \bar{a}_3]$.

"not element of", or "not in the set"!

We now consider a problem where a particular right-hand side (rhs) or target vector is not given, but questions are asked based only on the coefficient matrix. To demonstrate, we will first solve the problem for a generic rhs vector $\bar{b}$, and then demonstrate how to answer the question without using any $\bar{b}$ vector.

Prob 18, pg 48

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}.$$ Do the columns of B span $\mathbb{R}^4$?

Reword: Can you write any vector $\bar{b} \in \mathbb{R}^4$ as a linear combination of columns of B?

Let $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. Is $\bar{b} \in \text{span}\{\text{columns of } B\}$?

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 2 & b_1 \\ 0 & 1 & 1 & -5 & b_2 \\ 1 & 2 & -3 & 7 & b_3 \\ -2 & -8 & 2 & -1 & b_4 \end{array}\right] \xrightarrow[R_4+2R_1]{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 2 & b_1 \\ 0 & 1 & 1 & -5 & b_2 \\ 0 & -1 & -1 & 5 & b_3-b_1 \\ 0 & -2 & -2 & 3 & b_4+2b_1 \end{array}\right] \xrightarrow[R_4+2R_2]{R_3+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 2 & b_1 \\ 0 & 1 & 1 & -5 & b_2 \\ 0 & 0 & 0 & 0 & b_3-b_1+b_2 \\ 0 & 0 & 0 & -7 & b_4+2b_1+2b_2 \end{array}\right]$$

Need $b_3 - b_1 + b_2 = 0$ for the system to be consistent.

So, not every $\bar{b} \in \mathbb{R}^4$ belongs to span{columns of B}.

The set of all vectors in span{columns of B} is the vectors $\bar{b}$ for which $b_3 - b_1 + b_2 = 0$, i.e., $b_1 = b_2 + b_3$.

<u>Result</u> In general, for $A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$ with $\bar{a}_j \in \mathbb{R}^m$, $\text{span}\{\bar{a}_1, \ldots, \bar{a}_n\} = \mathbb{R}^m$ if and only if (echelon form of) A has a **pivot in every row**.

<u>Prob 22, pg 48</u> $\bar{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$. Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span $\mathbb{R}^3$?

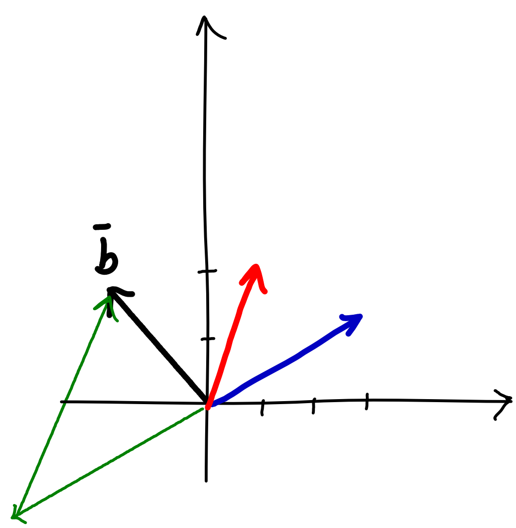
$$\begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \xrightarrow{R_3 \rightleftharpoons R_1} \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

There is a pivot in every row. Hence $\text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \mathbb{R}^3$.

Similarly, $\text{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \mathbb{R}^2$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 0 & -5 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 \rightleftharpoons R_2} \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$$

pivot in every row



Can write any vector $\bar{b} \in \mathbb{R}^2$ as a linear combination of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Note: The echelon form of A can have a pivot in every row with or without having non-pivot columns. Hence we cannot answer questions about uniqueness of solution(s) for every $\bar{b}$ just based on the fact that it has a pivot in every row.

Prob 30, pg 49 Construct a 3x3 **nonzero** matrix, not in echelon form whose columns do not span $\mathbb{R}^3$. Justify your answer.

every entry is $\neq 0$.

The matrix should not have a pivot in every row.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 3 & 4 \end{bmatrix}$$

cannot have a pivot. To make sure, we go to the echelon form here.

$$\xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \rightleftarrows R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

R_1 and R_2 are identical here!

Alternatively, how to find a non-zero matrix whose columns do span $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns here do span $\mathbb{R}^3$, as there is a pivot in every row. Now, use EROs to convert the zero entries to non-zero.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Properties of matrix-vector multiplication

$A \in \underbrace{\mathbb{R}^{m\times n}}_{\text{set of all } m\times n \text{ matrices with real entries}}$, $\bar{u}, \bar{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$.

(a) $A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$.

(b) $A(c\bar{u}) = c(A\bar{u})$.

<u>Proof for (a)</u> Let $A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$ ← $m\times n$ matrix, where $\bar{a}_j \in \mathbb{R}^m$, and $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

We assume vector-scalar multiplication and vector addition properties here.

$$\bar{u}+\bar{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}.$$

$$A(\bar{u}+\bar{v}) = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} = \bar{a}_1(u_1+v_1) + \bar{a}_2(u_2+v_2) + \cdots + \bar{a}_n(u_n+v_n)$$

$$= (\bar{a}_1 u_1 + \bar{a}_1 v_1) + (\bar{a}_2 u_2 + \bar{a}_2 v_2) + \cdots + (\bar{a}_n u_n + \bar{a}_n v_n)$$

as $(c+d)\bar{u} = c\bar{u} + d\bar{w}$ for $c, d \in \mathbb{R}$, $\bar{w} \in \mathbb{R}^n$

$$= (\bar{a}_1 u_1 + \bar{a}_2 u_2 + \cdots + \bar{a}_n u_n) + (\bar{a}_1 v_1 + \bar{a}_2 v_2 + \cdots + \bar{a}_n v_n)$$

as $\bar{w}+\bar{v} = \bar{v}+\bar{w}$ for $\bar{u}, \bar{v} \in \mathbb{R}^n$

$$= \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= A\bar{u} + A\bar{v}.$$