MATH 230 - Lecture 6 (01/27/2011)

Notation for replacement EROs

$$\begin{bmatrix} & \\ & \end{bmatrix} \xrightarrow[3R_2+R_1]{R_1+3R_2}$$

R_1+3R_2: Replace **Row 1** by the sum of itself and $3\times$(Row 2).

$3R_2+R_1$: Scale **Row 2** by 3 and then replace it by the sum of the scaled version and Row 1.

These two are NOT the same!

Of course, they are both valid EROs.

Another somewhat confusing notation – combining EROs:

$\xrightarrow[R_2\times(2)]{R_1+3R_2}$ We first add $3\times$(Row 2) to Row 1, and then scale (Row 2) by 2.

$A\bar{x}=\bar{b}$ has a solution for every $\bar{b}\in\mathbb{R}^m$ if there is a pivot in every row of $A\in\mathbb{R}^{m\times n}$, i.e., it has m pivots.

$\mathbb{R}^{m\times n}$: Set of all $m\times n$ real matrices

We can answer general questions just based on A, without a particular $\bar{b}$ given. For instance, recall Butch's dilemma between partying and tutoring.

$$\begin{aligned} x_1+x_2&=h \\ 8x_1+16x_2&=d \end{aligned} \quad \left\{\begin{aligned} h &= \text{\# hrs available} \\ d &= \text{\# dollars available} \end{aligned}\right\}$$

Qn. Can Butch balance partying & tutoring for any h & d?

YES! $\begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix} \xrightarrow{R_2-8R_1} \begin{bmatrix} 1 & 1 \\ 0 & 8 \end{bmatrix}$ pivot in every row.

Homogeneous System of Linear Equations (Section 1.5)

$A\bar{x} = \bar{0}$ is a homogeneous system

all right-hand side entries are zero.

$\bar{x} = \bar{0}$ is always a solution, and hence is called the **trivial solution**. We are interested in finding more interesting solutions.

A non-zero (i.e, at least one entry $\neq 0$) $\bar{x}$ which is a solution is called a **non-trivial solution**.

Prob 2. pg 55

$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

Does this system have a non-trivial solution?

The rhs column always remains at zero, and hence we can ignore it from the augmented matrix.

$$\begin{bmatrix} 1 & -3 & 7 \\ -2 & 1 & -4 \\ 1 & 2 & 9 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & -3 & 7 \\ 0 & -5 & 10 \\ 0 & 5 & 2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -3 & 7 \\ 0 & -5 & 10 \\ 0 & 0 & 12 \end{bmatrix}$$

pivot in every row, and no free variables.

We can use more EROs to go to the reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence the system has a unique solution $x_1=0, x_2=0, x_3=0$, which is the trivial solution. So it does not have any non-trivial solutions.

Note: We need **free variable(s)** to get **non-trivial solutions** to $A\bar{x}=\bar{0}$.

Prob 3, pg 55

Describe the solution set.

$$-3x_1+5x_2-7x_3=0$$
$$-6x_1+7x_2+x_3=0$$

$A\bar{x}=\bar{0}$ with $A=\begin{bmatrix} -3 & 5 & -7 \\ -6 & 7 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -3 & 5 & -7 \\ -6 & 7 & 1 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} -3 & 5 & -7 \\ 0 & -3 & 15 \end{bmatrix} \xrightarrow[R_2\times -1/3]{R_1\times -\frac{1}{3}} \begin{bmatrix} 1 & -5/3 & 7/3 \\ 0 & 1 & -5 \end{bmatrix} \xrightarrow{R_1+5/3R_2}$$

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \end{bmatrix}$$

x_3 is free

$$x_1 \quad 6x_3=0$$
$$x_2-5x_3=0$$

$$\left.\begin{array}{l} x_1=6x_3 \\ x_2=5x_3 \\ x_3 \text{ free} \end{array}\right\} \quad \bar{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} x_3$$

x_3 free.

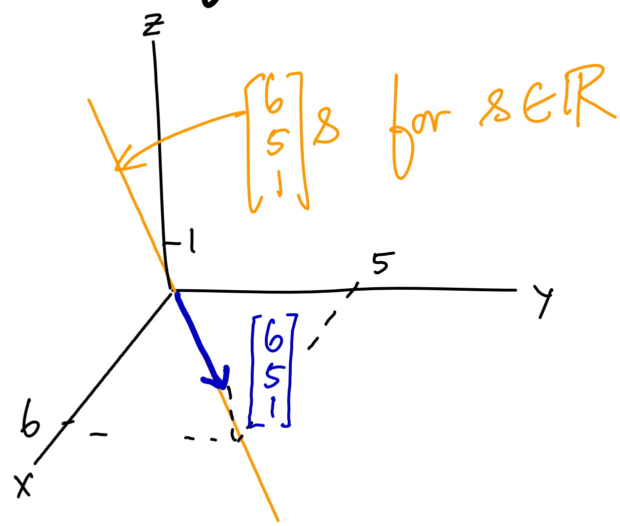
$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} x_3$$

We can write these solutions as

$$\bar{x} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} s$$ for a parameter s, whose value can be freely set.

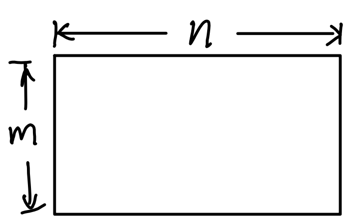
(vector) x (parameter)

This is the **parametric vector form** of all solutions to $A\bar{x} = \bar{0}$



Setting $s=0$ gives the trivial solution.

<u>Detour</u>: Will we always have non-trivial solutions for $A\bar{x} = \bar{0}$, when $n > m$?



$m < n$

The max # pivots A can have is m here. Hence $A\bar{x} = \bar{0}$ has at least one free variable. So there are non-trivial solutions.

Back to the problem ...

Let us consider a non-homogeneous system corresponding to the homogeneous system we just solved.

$$-3x_1+5x_2-7x_3 = 3$$
$$-6x_1+7x_2+x_3 = 7$$

Note: In $A\bar{x}=\bar{b}$, if at least one entry in $\bar{b}$ is $\neq 0$, the system is non-homogeneous.

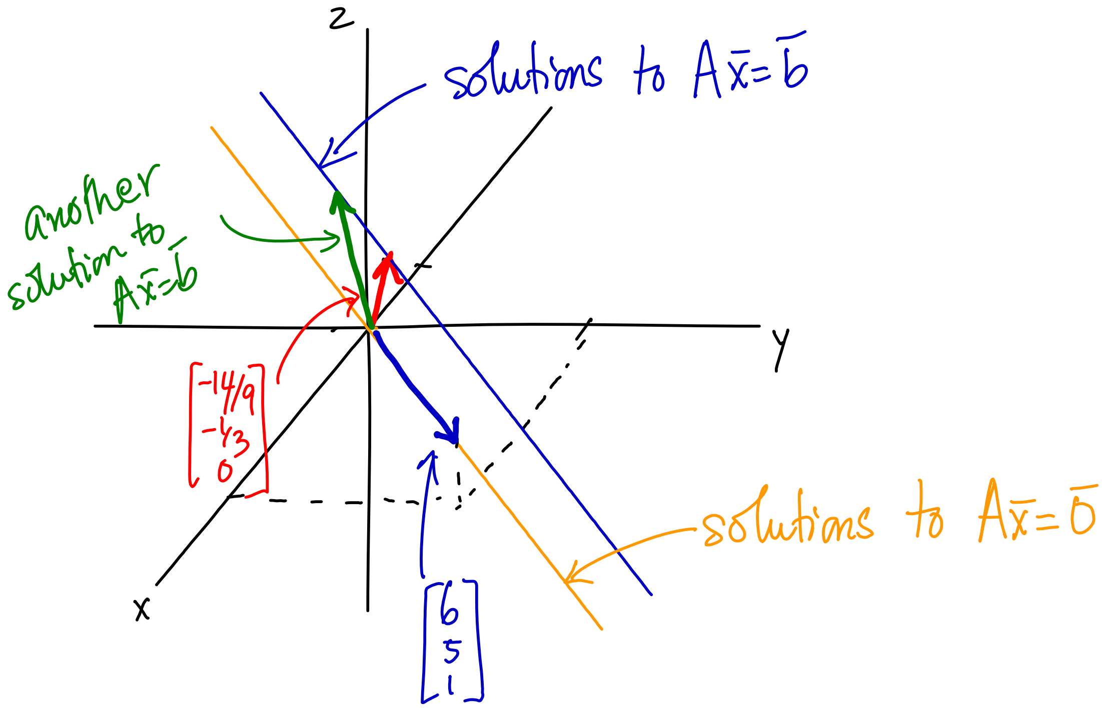
$$\left[\begin{array}{ccc|c} -3 & 5 & -7 & 3 \\ -6 & 7 & 1 & 7 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} -3 & 5 & -7 & 3 \\ 0 & -3 & 15 & 1 \end{array}\right] \xrightarrow[R_2\times -1/3]{R_1\times -1/3} \left[\begin{array}{ccc|c} 1 & -5/3 & 7/3 & -1 \\ 0 & 1 & -5 & -1/3 \end{array}\right]$$

$$\xrightarrow{R_1+5/3 R_2} \left[\begin{array}{ccc|c} 1 & 0 & -6 & -14/9 \\ 0 & 1 & -5 & -1/3 \end{array}\right] \qquad \begin{array}{l} x_1 = 6x_3 - 14/9 \\ x_2 = 5x_3 - 1/3, \quad x_3 \text{ free} \end{array}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{matrix} 6x_3 - 14/9 \\ 5x_3 - 1/3 \\ x_3 \end{matrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -14/9 \\ -1/3 \\ 0 \end{bmatrix} \quad \text{or}$$

$$\bar{x} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} s + \begin{bmatrix} -14/9 \\ -1/3 \\ 0 \end{bmatrix}, \quad s \in \mathbb{R}$$ is the parametric vector form of the solution.

Note that the solutions to the non-homogeneous system is obtained by adding the vector $\begin{bmatrix} -14/9 \\ -1/3 \\ 0 \end{bmatrix}$ to the solutions of the homogeneous system.



Result (Theorem 6 in pg 53). Let $\bar{p}$ be one particular solution to $A\bar{x}=\bar{b}$, i.e., $A\bar{p}=\bar{b}$. Then all solutions to $A\bar{x}=\bar{b}$ are given by $\bar{x}=\bar{p}+\bar{v}_h$, where $\bar{v}_h$ is a solution to $A\bar{x}=\bar{0}$.

Illustration

$\bar{q}$ is another particular solution to $A\bar{x}=\bar{b}$.