# MATH 230 - Lecture 9 (02/08/2011)

## Special cases of LI/LD vectors

② 2 vectors (seen in last lecture).

e.g., $\bar{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$ are LD, as $\bar{v}_2 = 3\bar{v}_1$.

③ If the set of vectors $\{\bar{v}_1, \ldots, \bar{v}_n\}$ contains the zero vector, then it is LD. (Theorem 9, pg 69, DL-LAA).

Say, $\bar{v}_1 = \bar{0}$ (zero vector). Then

$$c_1\bar{v}_1 + 0\cdot\bar{v}_2 + \cdots + 0\cdot\bar{v}_n = \bar{0} \quad \text{for } c_1 \neq 0.$$ As such,

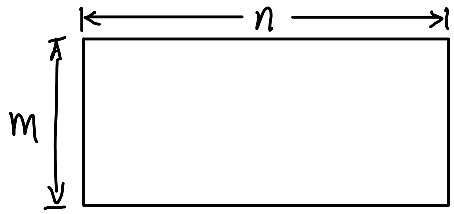
$\bar{x} = \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a non-trivial solution to $A\bar{x} = \bar{0}$ with

$A = [\bar{v}_1\ \bar{v}_2 \cdots \bar{v}_n]$.

④ $\{\bar{v}_1, \ldots, \bar{v}_n\}$ with $\bar{v}_j \in \mathbb{R}^m$ and $n > m$ is LD.

If there are more vectors than the # entries in each of them, the set is LD. (Theorem 8, pg 69, DL-LAA)

For $A = [\bar{v}_1 \ \bar{v}_2 \cdots \bar{v}_n]$, for $\bar{v}_j \in \mathbb{R}^m$ with $m < n$, the matrix looks like




. It must have at least one free variable, as the max. # pivots it can have is $m$.

<u>Prob 17, pg 71</u> Is the set of vectors LD? Justify.

$$\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$$

$\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3$

This set of vectors is LD, as it contains the zero vector.

Note that $\{\bar{v}_1\}$ as well as $\{\bar{v}_1, \bar{v}_3\}$ here are LI.

$\{\bar{v}_1, \bar{v}_3\}$ : $\begin{bmatrix} 3 & -6 \\ 5 & 5 \\ -1 & 4 \end{bmatrix} \xrightarrow[R_2 + 5R_3]{R_1 + 3R_3} \begin{bmatrix} 0 & 6 \\ 0 & 25 \\ -1 & 4 \end{bmatrix}$ No free variables. Hence $\{\bar{v}_1, \bar{v}_3\}$ is LI.

But $\{\bar{v}_1, \bar{v}_2\}$ and $\{\bar{v}_2, \bar{v}_3\}$ are both LD, as $\bar{v}_2 = \bar{0}$.

But if a set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LI, then all its subsets (or sub collections of vectors) must be LI.

## Theorem 7 (DL-LAA page 68)

$S=\{\bar{v}_1,\dots,\bar{v}_n\}$ of 2 or more vectors $(n\geq 2)$ in $\mathbb{R}^m$ is LD if and only if one vector in $S$ is a linear combination of the other vectors. In fact, if $S$ is LD and $\bar{v}_1\neq\bar{0}$, then some $\bar{v}_j$, $j>1$ is a linear combination of the preceding vectors $\bar{v}_1,\bar{v}_2,\dots,\bar{v}_{j-1}$.

e.g., $\bar{v}_3=c_1\bar{v}_1+c_2\bar{v}_2$ or
$\bar{v}_5=c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3+c_4\bar{v}_4$, etc.!

### Part of proof (Part 2)

$S=\{\bar{v}_1,\dots,\bar{v}_n\}$ is LD

iff ← (if and only if) or $\Longleftrightarrow$

$c_1\bar{v}_1+\dots+c_n\bar{v}_n=\bar{0}$ for $\bar{c}=\begin{bmatrix}c_1\\ \vdots\\ \vdots\\ c_n\end{bmatrix}\neq\bar{0}$ (at least one $c_j\neq 0$)

Given that $\bar{v}_1\neq 0$, we cannot have $\bar{c}=\begin{bmatrix}c_1\\0\\0\\ \vdots\\0\end{bmatrix}$ as a possible linear combination that gives the zero vector.

In general, we can have $c_1\neq 0$, $c_2\neq 0,\dots,c_j\neq 0$, and $c_{j+1}=0$, $c_{j+2}=0,\dots,c_n=0$ for a combination that works, for some $j>1$.

We get $c_1\bar{v}_1 + \cdots + c_j\bar{v}_j + 0.\bar{v}_{j+1} + \cdots + 0.\bar{v}_n = \bar{0}$

$\neq 0 \;\ldots \neq 0 \quad \neq 0$

$$\Rightarrow \quad c_j\bar{v}_j = -c_1\bar{v}_1 - c_2\bar{v}_2 - \cdots - c_{j-1}\bar{v}_{j-1}$$

("implies")

As $c_j \neq 0$, dividing by $c_j$ gives

$$\bar{v}_j = \left(\frac{-c_1}{c_j}\right)\bar{v}_1 + \left(\frac{-c_2}{c_j}\right)\bar{v}_2 \cdots + \left(\frac{-c_{j-1}}{c_j}\right)\bar{v}_{j-1}$$

Hence $\bar{v}_j$ can be written as a combination of the vectors preceeding it.

<u>Prob 6, Pg 71</u>

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}.$$ Are the columns of A LI?

Check if $A\bar{x} = \bar{0}$ has non-trivial solutions or not.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \xrightarrow[R_4-5R_3]{R_1+4R_3} \begin{bmatrix} 0 & -3 & 12 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 0 & 4 & -9 \end{bmatrix} \xrightarrow[R_4+4R_2]{R_1-3R_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix} \xrightarrow[R_4\to R_3]{R_3\to R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column has a pivot, so no free variables. Hence the columns of A are LI.

## Prob 36 Pg 72 True/False "with more justification"

If $\bar{v}_1, \dots, \bar{v}_4$ are in $\mathbb{R}^4$, and $\bar{v}_3$ is not a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_4$, then $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is L.I.

The statement is false, as we could have $\bar{v}_4$ being a linear combination of $\bar{v}_1$ and $\bar{v}_2$. Or, $\bar{v}_1$, for instance, could be the zero vector.

If $\underbrace{\bar{v}_4 = c_1\bar{v}_1 + c_2\bar{v}_2}$, then $\bar{x} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ -1 \end{bmatrix}$ is a non-trivial solution to the system $A\bar{x} = \bar{0}$, where $A = [\bar{v}_1\ \bar{v}_2\ \bar{v}_3\ \bar{v}_4]$.

$$c_1\bar{v}_1 + c_2\bar{v}_2 + 0\bar{v}_3 - \bar{v}_4 = \bar{0}$$

## Linear Transformation (Section 1.8)

"mappings"

$A\bar{x} = \bar{b}$

"Hit" $\bar{x}$ with $A$ to get $\bar{b}$

$A$ "maps" $\bar{x}$ to $\bar{b}$

When $\bar{x}$ is "input" to $A$, $\bar{b}$ is returned.

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$A\bar{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \times 3 + \begin{bmatrix} -5 \\ 7 \end{bmatrix} \times 1 + \begin{bmatrix} -7 \\ 5 \end{bmatrix} \times 0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$