Calculus III (Math 273, Section 2) – Fall 2014
Final Exam

- There are **ten** problems and **six** pages in this exam.
- Show all work, and provide appropriate **justifications** where required.
- Calculators, cell phones, laptops, or any other electronic devices are **not** allowed.
- Good luck!

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1. **(8)** Find the average value of $f(x, y) = xy$ over the quarter-circular disk $x^2 + y^2 \leq 1$ in the first quadrant.

2. **(10)** Sketch the region of integration of the following sum of polar integrals. Then convert the sum of polar integrals to a Cartesian integral, or a sum of Cartesian integrals. Do **not** evaluate the Cartesian integral(s).

$$\int_0^{\pi/6} \int_1^{2\sqrt{3}\sec\theta} r^6 \sin^2\theta \cos\theta \, dr d\theta \; + \; \int_{\pi/6}^{\pi/2} \int_1^{2\csc\theta} r^6 \sin^2\theta \cos\theta \, dr d\theta.$$

3. **(8)** Find the area of the region in the xy-plane bounded by the lines $y = -x + 1$, $y = x - 3$, and the curve $y = \sqrt{x - 1}$. It would help to sketch the region.

4. **(8)** Evaluate the following integral by changing to polar coordinates.

$$\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} \, dx dy.$$

5. **(8)** Integrate $f(x, y, z) = \sqrt{x^2 + z^2}$ over the circle $\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}$, $0 \leq t \leq 2\pi$.

6. **(8)** Find the line integral of $f(x, y) = x^2 / y^{4/3}$ over the curve $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, $1 \leq t < 2$.

7. **(8)** Find the work done by the vector field $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j} + \mathbf{k}$ when moving from $A = (0, 0, 0)$ to $B = (1, 1, 1)$ along the straight line connecting the two points.

8. **(20)** Find the flux and circulation by evaluating the line integrals (in Part 8a). Then compute these quantities using Green's theorem (in Part 8b).

   (a) Find the flux and circulation of the vector field $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$ across and around the closed semicircular path consisting of the semicircular arch of radius a lying above the x-axis going from $(a, 0)$ to $(-a, 0)$ in the counterclockwise direction, followed by the line segment from $(-a, 0)$ to $(a, 0)$. You could use the following integral:

   $$\int (\sin^3 x + \cos^3 x)\, dx = \frac{1}{12}\Big(9\sin x + \sin(3x) - 9\cos x + \cos(3x)\Big) + \text{ constant.}$$

   (b) For $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ and a piecewise smooth closed curve C which bounds the region R, two forms of Green's theorem (in 2D) can be specified as follows. Here, $\mathbf{T}$ is the unit tangent and $\mathbf{n}$ is the unit normal vector at each point on C.

   $$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \quad \text{and} \quad \oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

   Find the circulation and flux for the field $\mathbf{F}$ and closed curve C given in Part 8a by evaluating the corresponding double integrals specified by Green's theorem.

9. **(12)** Decide whether each of the following statements is *True* or *False*. **Justify** your answer.

   (a) The average height of a surface $z = f(x, y)$ above a region R in the xy-plane cannot be computed using polar coordinates.

   (b) The order of integration in polar coordinates, which is first r and then θ, cannot be reversed without changing the integral.

   (c) The line integral of a vector field along a curve $\mathbf{r}(t)$ depends only on the magnitude of $\dfrac{d\mathbf{r}}{dt}$, and not on its direction.

   (d) The circulation of a vector field around two different closed curves C_1 and C_2 is the same when both C_1 and C_2 are unit circles.

10. **(10)** Similar to the definition given in Cartesian (x, y) coordinates, the average value of the function $f(r, \theta)$ over a region R in polar coordinates is given by

    $$\hat{f} = \frac{1}{\text{Area}(R)} \iint_R f(r, \theta)\, r\,dr\,d\theta.$$

    Using the above definition, find the average distance from the point $P(x, y)$ in the disk $x^2 + y^2 \le a^2$ to the origin.