MATH 273 – Lecture 1 (08/26/2014)

Calculus III - Section 2

I'm Bala Krishnamoorthy. Call me Bala.

See the syllabus on the course web page. All info and documents relevant for this course will be available from the web page. In particular, no paper versions will be handed out in class.

Functions of several variables (Section 13.1 from the book).

Def Let D be a set of n-tuples of real numbers $(x_1, \dots, x_n)$. A real-valued **function** f on D is a rule that assigns a unique real number

$$w = f(x_1, \dots, x_n)$$

to each element in D.

D is the **domain** of f. The set of real values w taken by f is its **range**.

f is a function of the **independent variables** $x_1, \dots, x_n$.

w is the **dependent variable**.

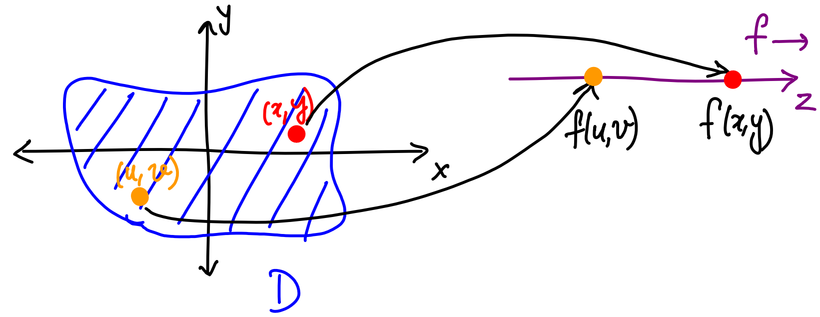
Another set of related terminology calls $x_1, \dots, x_n$ the **input variables**, and w the **output variable**.

One could think of "putting in" $x_1, x_2, \dots, x_n$ into the "box" that is f, and getting out w. For a given tuple $(x_1, \dots, x_n)$, the output w is unique.

Notation

In 2D, we write $z = f(x,y)$.
In 3D, we write $w = f(x,y,z)$.

Indeed, we will "draw" the 2D function in 3D using (x,y,z) coordinates!



Evaluating functions

$\longrightarrow$ plug in the values of x and y!

Prob 2b (pg 692). $f(x,y) = \sin(xy)$

(b) $f(\underset{x}{-3}, \underset{y}{\pi/12}) = \sin\left(-3 \times \frac{\pi}{12}\right) = \sin\left(\frac{-\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Domains and Ranges (of functions)

We extend the ideas used to define the domain of functions of a single variable to those of several variables. As in the 1D case, we look to avoid division by zero and negative values inside square roots.

Def The **domain** of f is the largest set of variable values that generate real numbers as output values.
The **range** of f is the set of all real output values.

Notice that both the domain and the range are sets.

Prob 5 (pg 692)

$f(x,y) = \sqrt{y-x-2}$. Find and describe the domain of f.

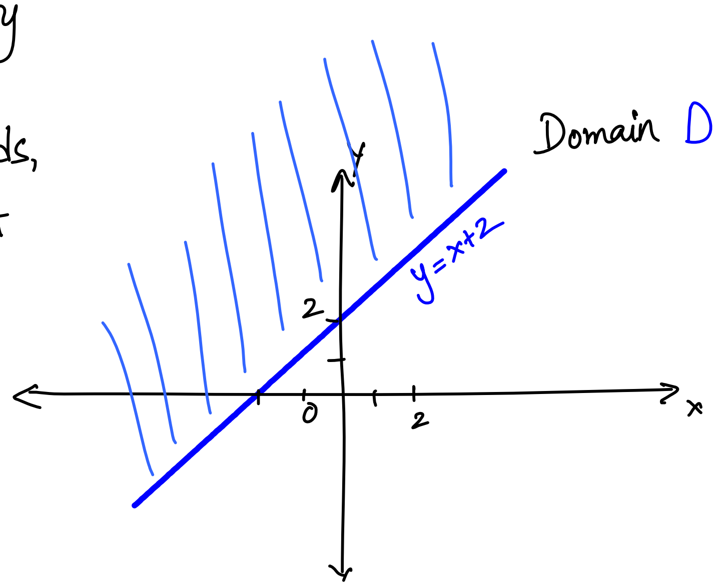
We need $y-x-2 \geq 0$, as the number inside the square root must be nonnegative. Indeed, the domain D of f here is the set of all pairs (x,y) of values in real 2D space that satisfy $y-x-2 \geq 0$.

Let's try to visualize D. For sets described by an inequality of this sort, we first plot the equation, and then pick the correct side of the line.

We first plot $y-x-2=0$, or $y=x+2$. This is the 45° line passing through $(0,2)$ and $(-2,0)$.

To pick the correct side, we try any point on the inequality to see if it holds. If it holds, that point is on the correct side. Since we could try any point, we try $(0,0)$ to keep things simple.

But, $0-0-2 \not\geq 0$. So $(0,0)$ is on the wrong side.



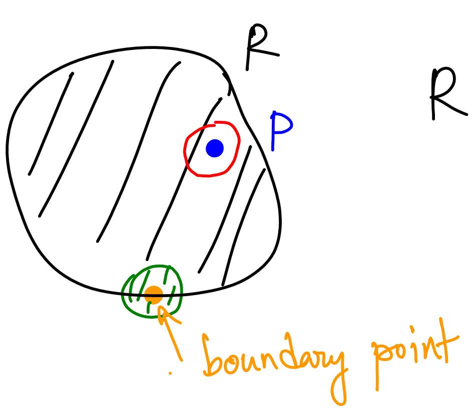
We now define properties of sets or regions that could describe the domain (and the function itself). In particular, we want to know if D is "open" or "closed", "bounded" (or finite) or "unbounded" etc.

Def A point $P(x,y)$ in a set R in the plane is an **interior point** of R if P is the center of a disc of some positive radius lying entirely in R.

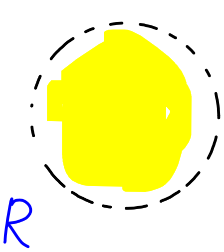
$P(x,y)$ is a **boundary point** of R if every disc centered at P has some points inside R and some outside. Notice that P itself need not be in R.

The set of all interior points : **interior** of R
The set of all boundary points : **boundary** of R.

If R contains only interior points, it is **open**.
If R contains its boundary, it is **closed**.



R here is bounded.



R here is open, as it does not contain boundary points.

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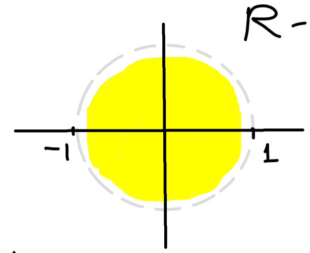
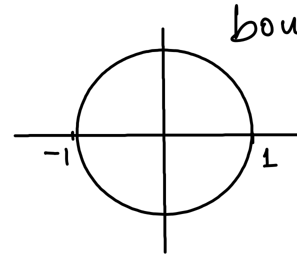
Examples of open and closed sets:

R - open set



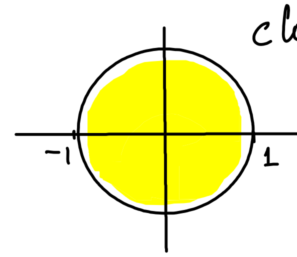
The unit disc without the points on the unit circle (shown in dotted gray here) is an open set, as all points in it are interior points.

boundary of R



Calling this set R, its boundary is the unit circle.

closed set

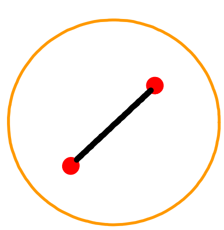


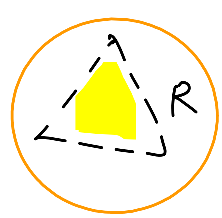
R and its boundary together form a closed set, which is the closed unit disc.

We introduce another concept that captures in some sense, how large a set or region is, i.e., whether it is limited or unlimited.

Def A region or set is **bounded** if it lies inside a disc of fixed radius. Else it is **unbounded**.

e.g.





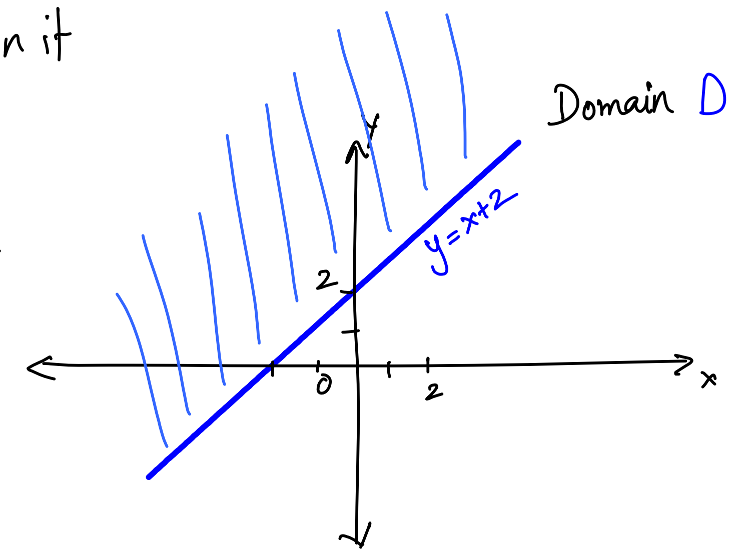
R is open, but is bounded.

A line segment, which is also closed.

Thus, a set can be bounded and at the same time be either open or closed. But notice that if a set is closed, it cannot be unbounded. Why?

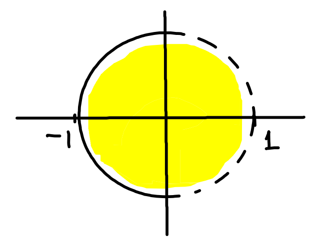
Going back to $f(x,y) = \sqrt{y-x-2}$ from Lecture 1, we can see that its domain D is unbounded.

D is not open here, as all points in it are not interior points. D is actually **closed** here. The line $y = x+2$ is indeed the boundary of D. Since D contains all its boundary points, it is closed.

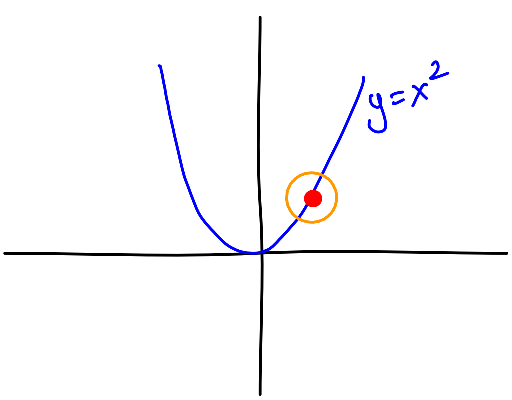


But some sets can be neither open nor closed!

Notice that the set made of the open unit disc and half its boundary is neither open nor closed.



Q. Is the parabola $y = x^2$ open or closed?



Every point on the parabola is a boundary point!
So, the set contains all its boundary points. Hence it is closed. Notice that there are no interior points in this set. The set is not open. It is unbounded, though.

How to visualize the range?

Level curves and Surfaces (of functions of two variables)

Def The set of points in the plane where $f(x,y)$ has constant value $f(x,y)=c$ is a **level curve** of f.
The set of all points $(x, y, f(x,y))$ is the **graph** of f, also called the **surface** of f, $z=f(x,y)$.

Prob 15, 21 (13.1)

$f(x,y)=xy$. Draw level curves for $f(x,y)=c$, $c=-9,-1,0,1,9$.

(a) Domain D?
(b) Range?
(c) Describe level curves.
(d) find the boundary of D?
(e) Is D open/closed/neither?
(f) Is D bounded?

$xy=c$

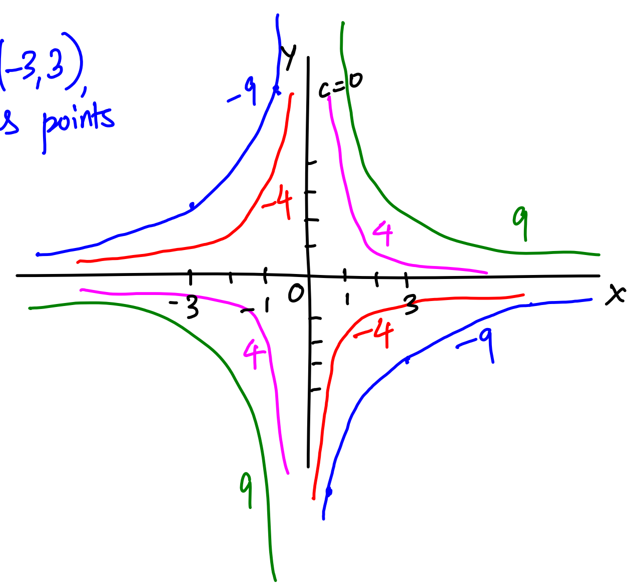
$c=0,\quad xy=0$

$c=-9\quad xy=-9,\ y=\dfrac{-9}{x}$

take $(-1,9)$, $(-3,3)$, and $(-9,1)$ as points

$c=-4\quad xy=-4,\ y=-\dfrac{4}{x}$

$c=9$, $c=4$



D: entire 2D plane

Range: all real numbers.

D is open and closed (boundary is empty).

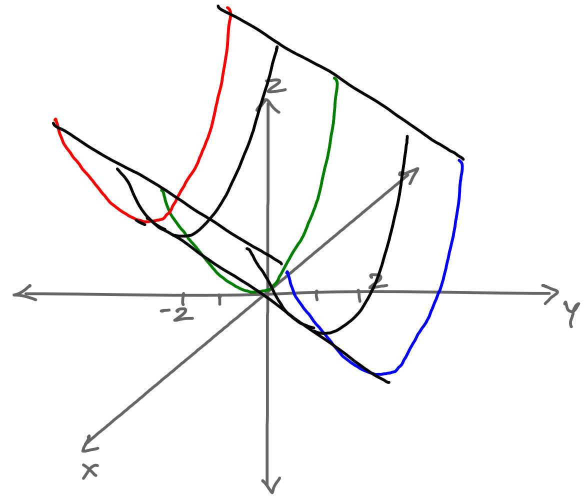
D is unbounded.

41 $f(x,y) = x^2 - y$

plot $(x, y, f(x,y))$.

$z = x^2 - y$

We know how to plot curves in 2D.
We plot the 3D surface by plotting several 2D curves together.



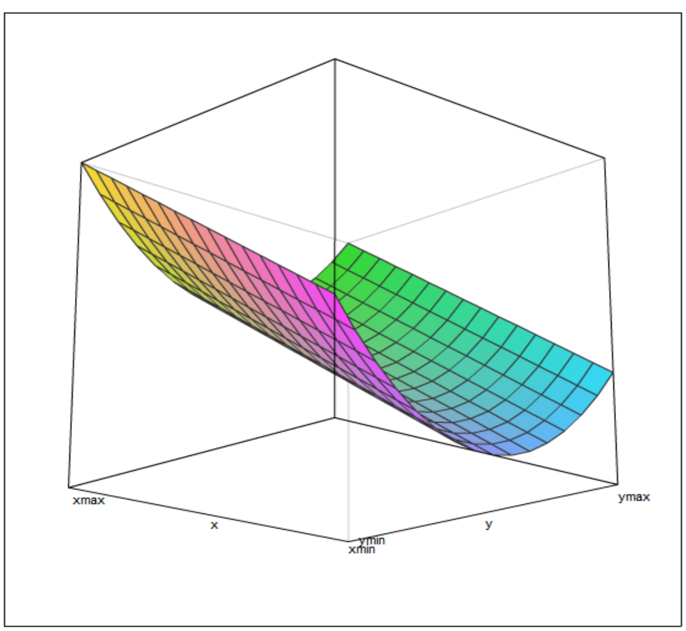
$y=0$: $z = x^2 \longrightarrow$ parabola through origin in xz-plane

$y=2$: $z = x^2 - 2 \rightarrow$ same parabola, but shifted down by 2

$y=-2$: $z = x^2 + 2 \longrightarrow$ Same parabola, but shifted up by 2

The "bowl" tilts at a 135° angle for y, as the y term is $-y$.

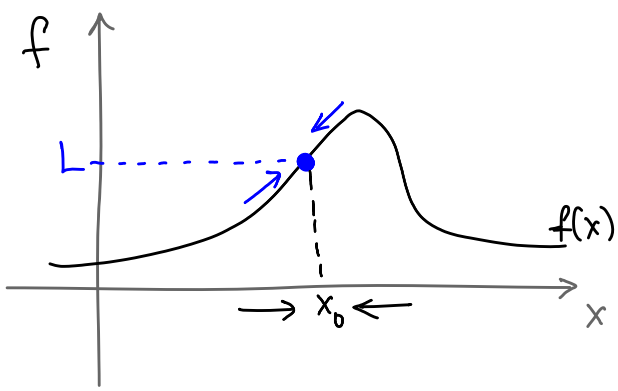
Here is a screenshot of the surface generated on the online 3D grapher (link given in the course web page)



Limits and continuity in higher dimensions (Section 13.2)

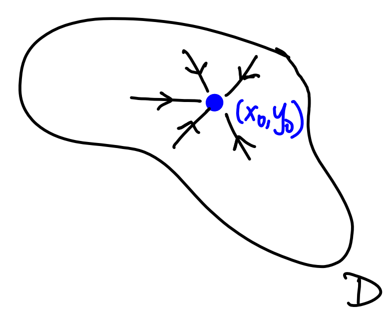
We will **not** cover this section in detail. There will be no homework assigned from this section as well.

Recall the concepts of limits and continuity defined in 1D.



Intuitively, as x approaches x_0 from left or right, if $f(x)$ tends to the same value L, say, then $\lim_{x \to x_0} f(x) = L$.

In 2D, (x,y) could approach some point (x_0, y_0) in infinitely many directions! But we extend the definition of limit the same way – $f(x,y)$ must tend to the same value L as (x,y) approaches (x_0, y_0) from every direction.



The idea of continuity is also extended from 1D to higher dimensions in a similar fashion. $f(x,y)$ is continuous at (x_0, y_0) if

1. $f(x_0, y_0)$ is defined,
2. $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$ exists, and
3. the limit in 2. is equal to $f(x_0, y_0)$.

We had defined derivatives of functions as limits (in 1D):

$$\frac{df(x)}{dx} = \lim_{x\to 0} \frac{f(x+h)-f(x)}{h}.$$

In multivariable calculus, we essentially apply single variable calculus on one variable at a time, keeping the other variables constant. We get partial derivatives in the latter case.

Partial Derivatives (Section 13.3)

Apply the definition of derivative in 1D to $f(x,y)$ one variable at a time, while keeping the other variable constant.

Def The **partial derivative** of $f(x,y)$ with respect to x at point (x_0,y_0) is

notice $y=y_0$ in both terms

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h},$$

also called "partial of f with respect to x", in short

provided that limit exists.

$\frac{\partial f}{\partial x}$ is also denoted $\frac{\partial f(x,y)}{\partial x}$, f_x, and $f_x(x,y)$.

The partial derivative with respect to y is defined analogously.

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h},$$ provided the limit exists.

To find $\frac{\partial f}{\partial x}$, we assume y is constant, and find $\frac{df}{dx}$.

Similarly, to find $\frac{\partial f}{\partial y}$, we assume x is constant, and find $\frac{df}{dy}$.

Prob 3 (13.3, page 711).

$f(x,y) = (x^2-1)(y+2)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = (y+2)\frac{d(x^2-1)}{dx} = (y+2)(2x) = 2x(y+2)$$

y term is constant here

$$\frac{\partial f}{\partial y} = (x^2-1)\frac{d}{dy}(y+2) = (x^2-1)(1) = x^2-1.$$

x-term is constant here

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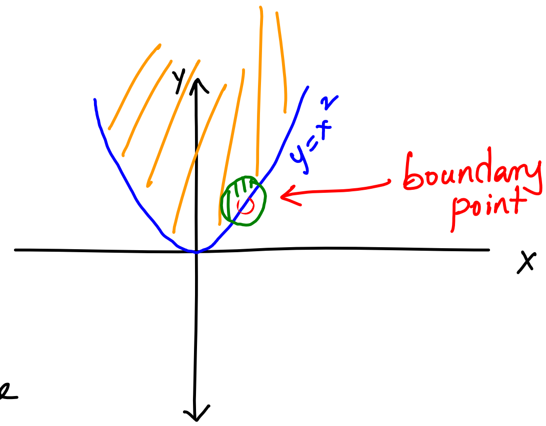
Open v/s closed sets – another example.

$R: \ y \geq x^2$

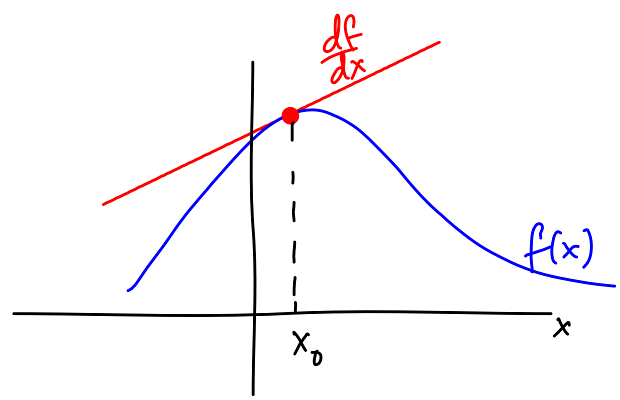


R is not open, as all points in R are not interior. Points on $y = x^2$ are not interior points.

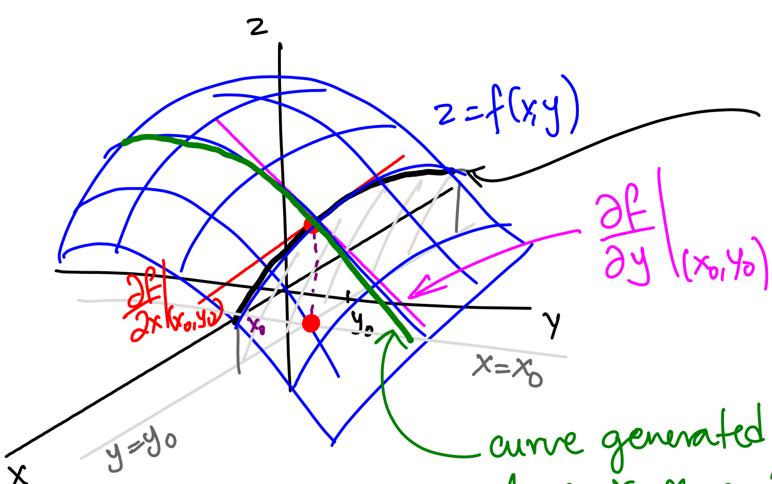
In fact, the points on $y = x^2$ are the boundary points. Since R contains its boundary points, R is closed.

Also, R is unbounded here.

In 1D, derivative of $f(x)$ at $x = x_0$ is the tangent to the curve $y = f(x)$ at $x = x_0$.



The intuition carries over to higher dimensions, one variable at a time.



curve generated by the intersection of the plane $y = y_0$ and the surface $z = f(x,y)$

curve generated by the intersection of the plane $x = x_0$ and the surface $z = f(x,y)$

Prob 27 (Section 13.3)

Find f_x, f_y, f_z where $f(x,y,z) = \sin^{-1}(xyz)$.

Recall that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f_x \text{ or } \frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(xyz)^2}}\,\frac{\partial}{\partial x}(xyz)$$

$$= \frac{yz}{\sqrt{1-x^2y^2z^2}}$$

$$f_y = \frac{1}{\sqrt{1-(xyz)^2}}\,\frac{\partial}{\partial y}(xyz) = \frac{xz}{\sqrt{1-x^2y^2z^2}}$$

In the exams, you will be given such complicated derivatives.

Similarly, $f_z = \dfrac{xy}{\sqrt{1-x^2y^2z^2}}$.

Prob 35 (Section 13.3)

$f(t,\alpha) = \cos(2\pi t - \alpha)$. Find the partial derivatives of f with respect to each variable.

$$f_t = \frac{\partial f}{\partial t} = -\sin(2\pi t-\alpha)\cdot\underbrace{\frac{\partial}{\partial t}(2\pi t-\alpha)}_{2\pi - 0} = -2\pi\sin(2\pi t-\alpha)$$

α is constant here

t is constant here

$$f_\alpha = \frac{\partial f}{\partial \alpha} = -\sin(2\pi t-\alpha)\cdot\underbrace{\frac{\partial}{\partial \alpha}(2\pi t-\alpha)}_{0-1} = -\sin(2\pi t-\alpha)\cdot(-1)$$

$$= \sin(2\pi t-\alpha).$$

This problem illustrates that we could use any symbol to represent variables, e.g., α and t here, and not just x, y, z, etc.

Implicit Differentiation

Rather than specifying $y = f(x)$ (i.e., the form of the function) explicitly, we could have an equation satisfied by x and y describing the function.

e.g., $xy + x^2 = 0$

To find $\frac{dy}{dx}$, we differentiate the equation with respect to x (both sides of '=').

$$\frac{d}{dx}\left(\underbrace{xy}_{\text{product}} + x^2\right) = \frac{d}{dx}(0) = 0$$

as $y = f(x)$ here

$1 \cdot y + x\frac{dy}{dx} + 2x = 0$, and solve for $\left(\frac{dy}{dx}\right)$

$x\left(\frac{dy}{dx}\right) = -(2x+y)$. So $\frac{dy}{dx} = \frac{-(2x+y)}{x}$.

We can do similar implicit partial differentiation in higher dimensions.

Prob 65 (Section 13.3) Find $\frac{\partial z}{\partial x}$ at the point $\underset{x,\,y,\,z}{(1,1,1)}$ given

$xy + z^3x - 2yz = 0$, $z = f(x,y)$, and that the partial derivative $\frac{\partial z}{\partial x}$ exists at $(1,1,1)$.

We differentiate the equation partially with respect to x. Thus, y is kept constant, but $z = f(x,y)$, and hence is a function of x.

$$\frac{\partial}{\partial x}\left(xy + \underbrace{z^3 x}_{\text{product}} - 2yz\right) = \frac{\partial}{\partial x}(0) = 0$$ y is constant here

$$y\frac{\partial}{\partial x}(x) + x\frac{\partial}{\partial x}(z^3) + z^3\frac{\partial}{\partial x}(x) - 2y\frac{\partial}{\partial x}(z) = 0$$

$$y + x\left(3z^2\frac{\partial z}{\partial x}\right) + z^3 - 2y\left(\frac{\partial z}{\partial x}\right) = 0$$

plug in $(x,y,z) = (1,1,1)$

$$1 + 1.3(1)^2\left(\frac{\partial z}{\partial x}\right) + (1)^3 - 2(1).\left(\frac{\partial z}{\partial x}\right) = 0$$

$$1 + 3\left(\frac{\partial z}{\partial x}\right) + 1 - 2\left(\frac{\partial z}{\partial x}\right) = 0$$ giving $\frac{\partial z}{\partial x} = -2$.

$$2 + (3-2)\left(\frac{\partial z}{\partial x}\right) = 0$$

Second order partial derivatives

If we differentiate $f(x,y)$ twice, we get

$\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$, or $f_{xx}, f_{yy}, f_{yx}, f_{xy}$.

Order of x, y : "start from inside" this is an easy to remember rule of thumb. The "inside" is where 'f' sits".

So, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ or $f_{yx} = (f_y)_x$

inside inside

Similarly, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, or $f_{xy} = (f_x)_y$.

Prob 42 (Section 13.3)

$f(x,y) = \sin xy$. find all second order partial derivatives of f.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos(xy)\cdot\frac{\partial(xy)}{\partial x}\right) = \frac{\partial}{\partial x}\Big(\underbrace{y\cos(xy)}_{\frac{\partial f}{\partial x}}\Big) = y\cdot\frac{\partial}{\partial x}(\cos(xy))$$

$$= y\cdot -\sin(xy)\frac{\partial}{\partial x}(xy) = -y^2\sin(xy).$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\cos(xy)\frac{\partial}{\partial y}(xy)\right) = \frac{\partial}{\partial y}\Big(\underbrace{x\cos(xy)}_{\frac{\partial f}{\partial y}}\Big) = x\cdot -\sin(xy)\frac{\partial}{\partial y}(xy)$$

$$= -x^2\sin(xy).$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\Big(\underbrace{x\cos(xy)}_{\text{product}}\Big) = x\cdot\frac{\partial}{\partial x}\cos(xy) + \overset{\frac{\partial}{\partial x}x}{1}\cdot\cos(xy)$$

$$= x\; -\sin(xy)\cdot y + \cos(xy)$$

$$= \cos(xy) - xy\sin(xy)$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\Big(\overbrace{y\cos(xy)}^{\frac{\partial f}{\partial x}}\Big) = 1\cdot\cos(xy) + y\cdot\frac{\partial}{\partial y}\cos(xy)$$

$$= \cos(xy) - xy\sin(xy)$$

Notice that $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$. Does this result always hold?

Under certain conditions, yes! More on this result in the next lecture.

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We noticed that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ in the last example from Lecture 3.

When does this result hold?

The mixed derivative theorem

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are all defined throughout an open region containing a point (a,b), and are all continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Applies to higher order derivatives as well, e.g., $f_{xxy} = f_{xyx} = f_{yxx}$

Prob 55 (Section 13.3) Which order will calculate f_{xy} faster – first x or first y? Answer without actually finding the derivatives.

(d) $f(x,y) = y + x^2 y + 4y^3 - \ln(y^2+1)$

x first, since x appears only in one term of f. To check, $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x.$

Prob 61 $f(x,y) = 2x + 3y - 4$. Find the slope of the line that is tangent to the surface $z = f(x,y)$ at $(\underset{x}{2}, \underset{y}{-1})$

(a) lying in the plane $x = 2$. $\frac{\partial f}{\partial y}$ at $(2,-1)$

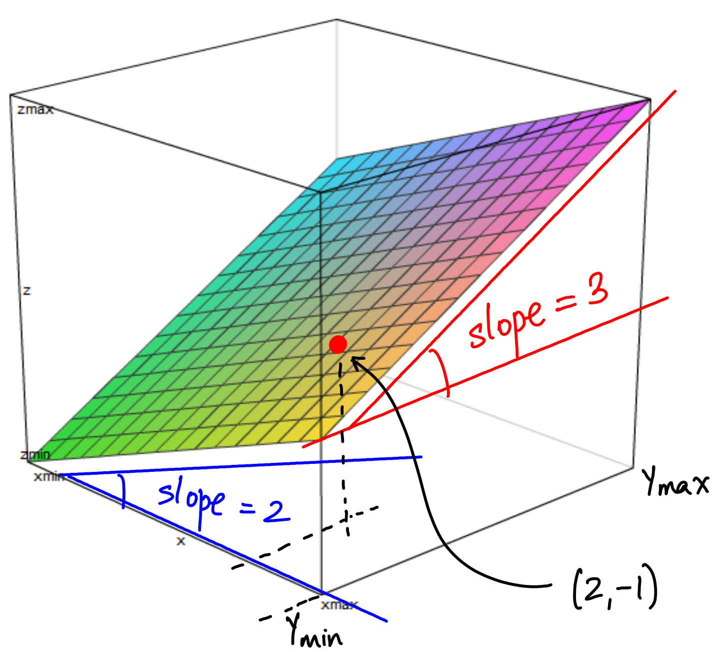
(b) lying in the plane $y = -1$. $\rightsquigarrow$ $\frac{\partial f}{\partial x}$ at $(2,-1)$

(a) $\frac{\partial f}{\partial y} = 3$. (b) $\frac{\partial f}{\partial x} = 2$

Recall from Lecture 3 that $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ is the tangent to the surface at (x_0,y_0) lying in the plane $y=y_0$. Notice that $y=y_0$ is a plane parallel to the xz-plane.

In this case, the surface is actually a plane – notice that all terms are linear – $2x$ and $3y$. Indeed, the partial derivatives in this case at $(2,-1)$ are the slopes of this plane at that point. Since it's a plane, these two slopes are the same irrespective of which point you are considering!

Here is the surface from online 3D grapher:



The Laplace Equation (still section 13.3)

The 3D Laplace equation is satisfied by the steady state temperature distribution in space. If $T = f(x,y,z)$, then

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad \text{or} \quad f_{xx} + f_{yy} + f_{zz} = 0$$

In 2D, we get $f_{xx} + f_{yy} = 0$.

Prob 73 (Section 13.3) Show f satisfies the Laplace equation.

$$f(x,y,z) = x^2 + y^2 - 2z^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x) = 2 \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(2y) = 2 \qquad \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(-4z) = -4$$

Hence $f_{xx} + f_{yy} + f_{zz} = 2 + 2 + (-4) = 0.$

Prob 78 $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$. Show f satisfies the Laplace equation.

Recall: $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$$f_{xx} = \frac{\partial}{\partial x}\left[\frac{1}{\left(1+\frac{x^2}{y^2}\right)}\frac{y^2}{y^2}\cdot\frac{1}{y}\right] = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{y\cdot(-1)\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

y is constant

$$y\frac{\partial}{\partial x}\left((x^2+y^2)^{-1}\right) = y\cdot(-1)(x^2+y^2)^{-2}\cdot 2x$$

$$f_{yy} = \frac{\partial}{\partial y}\left[\frac{1}{\left(1+\frac{x^2}{y^2}\right)}\cdot x\,\frac{-1}{y^2}\right] = \frac{\partial}{\partial y}\left(\frac{-x}{x^2+y^2}\right) = -x\frac{\partial}{\partial y}\left[(x^2+y^2)^{-1}\right]$$

$$\left(1+\frac{x^2}{y^2}\right)y^2$$

$$= -x(-1)(x^2+y^2)^{(-2)}\cdot 2y$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

So, $f_{xx} + f_{yy} = 0.$

The Chain Rule (Section 13.4)

In 1D, $w=f(x)$ and $x=g(t)$, and if f is differentiable w.r.t (with respect to) x, and g is differentiable w.r.t t, then w is differentiable w.r.t t, and

$$\frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt}$$ (Chain rule for one independent variable t and one intermediate variable x)

We generalize to higher dimensions, i.e., we consider more than one independent variable as well as more than one intermediate variable.

Theorem 5. Chain rule for one independent variable and two intermediate variables.

$w=f(x,y)$ is a differentiable function of x and y, and $x=x(t)$ and $y=y(t)$ are differentiable functions of t, then $w=f(x(t),y(t))$ is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} \quad \text{or}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

Notice how the partial derivatives are appearing – for the function f of multiple intermediate variables. The derivatives w.r.t to the single independent variable t are not partial.

Prob 2 (13.4) $w = x^2+y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$

Express $\frac{dw}{dt}$ using (a) chain rule and (b) by first writing w as a function of t, and then finding $\frac{dw}{dt}$.

(a) Chain rule: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$

$x = \cos t + \sin t$

$$= (2x)\underbrace{[-\sin t + \cos t]}_{y} + (2y)\underbrace{[-\sin t - \cos t]}_{-(\cos t + \sin t) = -x}$$

$$= 2x[y] + 2y[-x] = 2xy - 2xy = 0.$$

(b) $w = x^2+y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2$

$$= 2\underbrace{(\cos^2 t + \sin^2 t)}_{1} = 2\cdot 1 = 2$$

$\frac{dw}{dt} = 0.$

MATH 273 – Lecture 5 (09/09/2014)

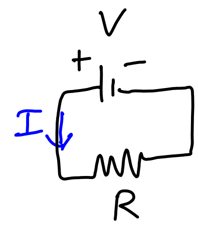
Next week (September 16 & 18): Class will meet in **VECS 125** ↳ via AMS Videoconferencing

Office hours on Skype (ID: **wsucomptopo**).

<u>Chain rule</u> $w = f(x,y)$, $x = x(t)$, $y = y(t)$, all functions differentiable.

Then $\dfrac{dw}{dt} = \dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt}$.

<u>Problem 41 (Section 13.4)</u>



V: voltage, I: current, R: resistance

$V = IR$

Voltage decreases with time, as the battery weakens.
Resistance increases with time as the resistor heats up.
How does the current change with time?

Find $\dfrac{dI}{dt}$ when $R = 600$ ohms (Ω), $I = 0.04$ amps, $\dfrac{dR}{dt} = 0.5$ ohms/sec, $\dfrac{dV}{dt} = -0.01$ volts/sec.

$V = IR$. Apply chain rule: $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial I} \cdot \dfrac{dI}{dt} + \dfrac{\partial V}{\partial R} \cdot \dfrac{dR}{dt}$

I, R are functions of time, and hence V is also one.

$$-0.01 = R \cdot \left(\frac{dI}{dt}\right) + I \left(\frac{dR}{dt}\right)$$

$\dfrac{\partial V}{\partial I} = \dfrac{\partial (IR)}{\partial I} = R \cdot 1 = R$.

i.e., $-0.01 = 600\left(\dfrac{dI}{dt}\right) + \underbrace{0.04(0.5)}_{0.02}$

This gives $-0.01 - 0.02 = 600\left(\dfrac{dI}{dt}\right)$. Hence $\dfrac{dI}{dt} = \dfrac{-0.03}{600} = -5 \times 10^{-5}$ amps/sec.

We can extend the chain rule to 3 intermediate variables.

Theorem 6 $w=f(x,y,z)$ is differentiable, and x,y,z are differentiable functions of t. Then w is a differentiable function of t, and

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Prob 6 $w=z-\sin xy$, $x=t$, $y=\ln t$, $z=e^{t-1}$.

Find $\frac{dw}{dt}$ using (a) chain rule, and (b) directly, and find $\frac{dw}{dt}$ at $t=1$ in each case.

(a) $$\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$=(0-y\cos xy)\cdot(1)+(0-x\cos xy)\left(\frac{1}{t}\right)+(1-0)\,e^{t-1}$$

at $t=1$, $x=t=1$, $y=\ln t=0$, $z=e^{1-1}=1$.

$$\left.\frac{dw}{dt}\right|_{t=1}=(-0\cancel{\cos 0})(1)+(-1\cdot\underbrace{\cos 0}_{1})\left(\frac{1}{1}\right)+1\cdot e^{1-1}$$

$$=0-1+1=0.$$

(b) $w=z-\sin xy=e^{t-1}-\sin(t\ln t)$

$$\frac{dw}{dt}=e^{t-1}-\cos(t\ln t)\cdot\underbrace{[1\cdot\ln t+\cancel{t}\cdot\tfrac{1}{\cancel{t}}]}$$

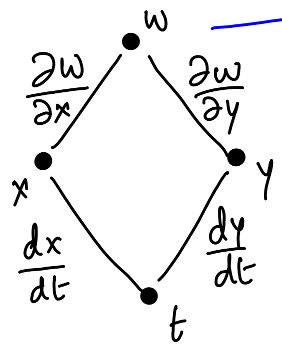
product rule $\frac{d(t)}{dt}\cdot\ln t+t\cdot\frac{d}{dt}(\ln t)$

$$\left.\frac{dw}{dt}\right|_{t=1}=e^{1-1}-\cos(1\cdot\underbrace{\ln 1}_{0})\,[1\cdot 0+1]=1-1\times 1=0.$$

Branch Diagrams

We present a pictorial way to easily decipher the expressions for derivatives using the chain rule. The idea is not to memorize the expressions by heart!

$w = f(x, y), \quad x = x(t), \quad y = y(t).$



→ dependent variable

→ intermediate variables

→ inpendent variable

Multiply the terms along each branch, and add to get the expression for $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \underbrace{\frac{\partial w}{\partial x} \cdot \frac{dx}{dt}}_{\text{left}} + \underbrace{\frac{\partial w}{\partial y} \cdot \frac{dy}{dt}}_{\text{right}}$$

There are three levels – one each of the dependent, intermediate, and the independent variables. We draw a point, or a "node", for each variable in the corresponding level. Then we draw a branch, or a line, from each node to the nodes in the next level.

If there is only one branch going down from a variable node, the corresponding derivative is the usual one, i.e., not a partial derivative. For instance, we have $\frac{dx}{dt}$ and $\frac{dy}{dt}$ above. When there are two or more branches going down, each branch gets a partial derivative term.

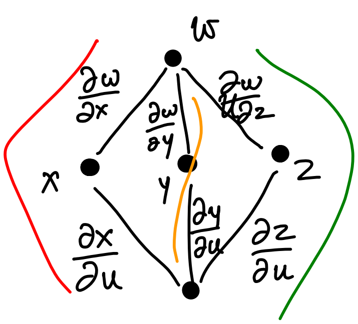
Prob 15 Draw branch diagram for $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ when

$$w = h(x,y,z), \quad x = f(u,v), \quad y = g(u,v), \quad z = k(u,v).$$

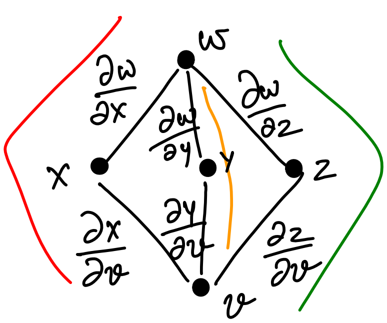
Note: This problem presents essentially the same result that is specified in Theorem 7. It is much easier to draw the branch diagrams to figure out the expressions for the partial derivatives!

Notice that w is the dependent variable, x,y,z are the intermediate variables, and u,v are the independent variables here.

Since there are two independent variables, w is ultimately a function of both of them (u,v here), and hence we get partial derivatives w.r.t to each of them. We draw separate branch diagrams for each of them as well.



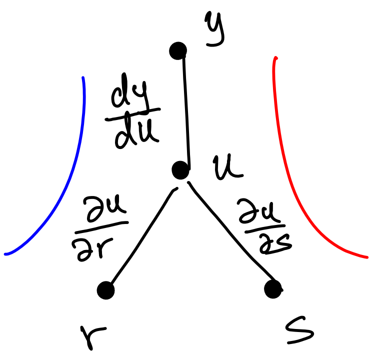
$$\frac{\partial w}{\partial u} = \underline{\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}} + \underline{\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}} + \underline{\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}}$$



$$\frac{\partial w}{\partial v} = \underline{\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}} + \underline{\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}} + \underline{\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}}$$

We do **not** add the two partial derivatives! We write them separately. Later on, in Section 13.6, we will learn how to consider linear combinations of such partial derivatives in order to define a tangent plane at a given point.

<u>Prob 20</u> Branch diagram $\frac{\partial y}{\partial r}$ for $y = f(u)$, $u = g(r, s)$



$$\rightarrow \frac{\partial y}{\partial r} = \frac{dy}{du} \cdot \frac{\partial u}{\partial r}$$

$$\rightarrow \frac{\partial y}{\partial s} = \frac{dy}{du} \cdot \frac{\partial u}{\partial s}$$

Here, we have 1 dependent variable y, 1 intermediate variable u, and 2 independent variables r and s. As such, y is ultimately a function of r and s, thus giving rise to two partial derivatives.

MATH 273 – Lecture 6 (09/11/2014)

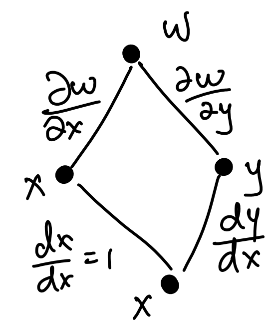
Chain rule (continued..)

<u>Implicit differentiation</u> $F(x,y)=0$ defines y as differentiable function of f. We differentiate both sides of the equation, and solve for $\frac{dy}{dx}$. We extend this idea to multiple variables and use the chain rule in the process.

<u>Prob 25</u> $x^3-2y^2+xy=0$ defines y as a differentiable function of x.
Find $\frac{dy}{dx}$ at $(1,1)$.

Let $w=F(x,y)=x^3-2y^2+xy$. Then $w=0$ is the given equation.

Differentiate w.r.t x (both sides):



$$\frac{dw}{dx}=\frac{\partial w}{\partial x}\underbrace{\frac{dx}{dx}}_{=1}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dx}=0$$

So,
$$(3x^2+y)\cdot 1+(-4y+x)\frac{dy}{dx}=0$$

Plug in $(x,y)=(1,1)$ to get

$$\underbrace{(3(1)^2+(1))}_{4}+\underbrace{(-4(1)+(1))}_{-3}\frac{dy}{dx}=0. \quad \text{So } \frac{dy}{dx}=\frac{4}{3}$$

<u>Prob 31</u>. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(\underset{x}{\pi},\underset{y}{\pi},\underset{z}{\pi})$ when
$F(x,y,z)=\sin(x+y)+\sin(y+z)+\sin(z+x)=0$ defines z implicitly as a differentiable function of x and y.

Note: x and y are independent variables, z is dependent on both x and y.

Differentiate partially w.r.t. to x $\quad F(x,y,z)=0$

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1}+\frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial x}}_{=0}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

So, $\left[\cos(x+y)\cdot 1+0+\cos(z+x)\cdot 1\right]\times 1+0+\left[\cos(y+z)+\cos(z+x)\right]\left(\frac{\partial z}{\partial x}\right)=0$

$$\left[\cos(x+y)+\cos(z+x)\right]+\left[\cos(y+z)+\cos(z+x)\right]\left(\frac{\partial z}{\partial x}\right)=0.$$

Plug in $(x,y,z)=(\pi,\pi,\pi)$ to get

If $\left[\cos(y+z)+\cos(z+x)\right]=0$ then $\frac{\partial z}{\partial x}$ is not defined

$$\left[\cos(2\pi)+\cos(2\pi)\right]+\underbrace{\left[\cos(2\pi)+\cos(2\pi)\right]}_{\neq 0}\left(\frac{\partial z}{\partial x}\right)=0$$

Hence, $\quad 2+2\left(\frac{\partial z}{\partial x}\right)=0$ giving $\frac{\partial z}{\partial x}=-1.$

Now, w.r.t. y, we get

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial y}}_{=0}+\frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial y}}_{=1}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y}=0$$

$$0+\left[\cos(x+y)\cdot 1+\cos(y+z)\cdot 1+0\right]\times 1+\left[\cos(y+z)+\cos(z+x)\right]\left(\frac{\partial z}{\partial y}\right)=0$$

Plug in $(x,y,z)=(\pi,\pi,\pi)$ to get

$$\left[\cos(2\pi)+\cos(2\pi)\right]+\left[\cos(2\pi)+\cos(2\pi)\right]\left(\frac{\partial z}{\partial y}\right)=0$$

$$2+2\left(\frac{\partial z}{\partial y}\right)=0, \quad \text{i.e.}\quad \frac{\partial z}{\partial y}=-1.$$

The book gives details of the implicit function theorem, which specifies when we can solve for $\left(\frac{\partial z}{\partial x}\right)$ in this fashion — as long as $\frac{\partial F}{\partial z}\neq 0$.

Prob 43 $f(u,v,w)$ is differentiable, and $u=x-y$, $v=y-z$, $w=z-x$.

Show that $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=0$.

f: dependent

u,v,w: intermediate

x,y,z: independent

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$
$$=f_u\cdot(1-0)+f_v\cdot(0-0)+f_w(0-1)$$
$$=f_u-f_w.$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}$$
$$=f_u\cdot(0-1)+f_v\cdot(1-0)+f_w(0-0)$$
$$=-f_u+f_v$$

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}$$
$$=f_u(0-0)+f_v(0-1)+f_w(1-0)$$
$$=-f_v+f_w$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=(\cancel{f_u}-\cancel{f_w})+(-\cancel{f_u}+\cancel{f_v})+(-\cancel{f_v}+\cancel{f_w})$$
$$=0.$$

Notice that we never used the exact form of $f(u,v,w)$ in this problem!

Direction derivative and Gradient Vector (Section 13.5)

We saw that $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ gives the slope of the line tangent to the surface $z=f(x,y)$ at (x_0,y_0) in the plane $y=y_0$. $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ was interpreted similarly. We now extend the idea of taking derivatives to arbitrary directions — and not just along x or y.

For a simple analogy, consider the function which gives the height of a mountain, $h=f(x,y)$. At any point on the mountain, the height may change differently in different directions.

Climbers on Mt. Rainier:

large negative slope — drop into crevasse!

tangent is steep (slope is large)

near zero slope - flat terrain (locally)

From where the climber is standing, the slope of the function (height) is very large looking to the right. It is negative looking to his left.

We define the directional derivative of a function f along a direction specified by the unit vector $\hat{u}$ at point $P_0 = (x_0, y_0)$ essentially in a similar fashion to how we have been defining derivatives so far using limits.

In 1D $$\left.\frac{df}{dx}\right|_{x_0} = \lim_{s \to 0} \frac{f(x_0 + s) - f(x_0)}{s}$$

$$f(x,y); \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{s \to 0} \frac{f(x_0+s, y_0) - f(x_0, y_0)}{s}$$

To generalize to considering derivatives in arbitrary directions, we think about $f(\cdot)$ as a function taking a vector as input.

Let $\bar{p} = \begin{bmatrix} x \\ y \end{bmatrix}$ (the vector with 2 components x and y, or, $\bar{p} = x\hat{i} + y\hat{j}$);

lower case letters with "bar" indicate vectors

and we consider the directional derivative of f at $\bar{P}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ in the direction $\hat{u}$ (unit vector).

$$\left. D_{\hat{u}} f \right|_{P_0} = \lim_{s \to 0} \frac{f(\bar{P}_0 + s\hat{u}) - f(P_0)}{s}$$

MATH 273 – Lecture 7 (09/16/2014)

Recall: directional derivative of $f(x,y)$ at $P_0(x_0,y_0)$ in the direction $\hat{u} = u_1\hat{i} + u_2\hat{j}$ ($\hat{u}$ is a unit vector, i.e., $\underbrace{\|\hat{u}\|}_{\text{length}} = 1$) is

$$\left(D_{\hat{u}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(P_0 + s\hat{u}) - f(P_0)}{s}$$

With $\bar{P}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = x_0\hat{i} + y_0\hat{j}$, we have

$$\bar{P}_0 + s\hat{u} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + s\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$= \left(x_0\hat{i} + y_0\hat{j}\right) + s\left(u_1\hat{i} + u_2\hat{j}\right)$$
$$= \left(x_0 + su_1\right)\hat{i} + \left(y_0 + su_2\right)\hat{j}$$

$$\left(D_{\hat{u}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(\overbrace{x_0+su_1}^{x}, \overbrace{y_0+su_2}^{y}) - f(x_0,y_0)}{s}$$

$x = x_0 + su_1 = x(s)$
$y = y_0 + su_2 = y(s)$

When $\hat{u} = \hat{i}$ (i.e., the x-direction), $\hat{u} = \underbrace{1}_{u_1}\cdot\hat{i} + \underbrace{0}_{u_2}\cdot\hat{j}$, then

$$\left(D_{\hat{i}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(x_0+s, y_0) - f(x_0,y_0)}{s} = \frac{\partial f}{\partial x}$$

Similarly, when $\hat{u} = \hat{j}$ (i.e., the y-direction), $\hat{u} = 0\cdot\hat{i} + 1\cdot\hat{j}$, then

$$\left(D_{\hat{j}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(x_0, y_0+s) - f(x_0,y_0)}{s} = \frac{\partial f}{\partial y}.$$

How to compute $\left(D_{\hat{u}}f\right)_{P_0}$?

We have $f(x,y)$ with x, y being functions of s. We apply chain rule to compute $f(x,y)$

$$x = x_0 + s u_1$$
$$y = y_0 + s u_2$$
$$\frac{dx}{ds} = u_1, \quad \frac{dy}{ds} = u_2$$
$$\hat{u} = u_1\hat{i} + u_2\hat{j} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\left.\frac{df}{ds}\right|_{\hat{u}, P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \underbrace{\frac{dx}{ds}}_{u_1} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \cdot \underbrace{\frac{dy}{ds}}_{u_2}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\hat{j}\right]}_{\text{gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\hat{i} + u_2\hat{j}\right]}_{\hat{u}} = \begin{bmatrix} \left(\frac{\partial f}{\partial x}\right)_{P_0} \\ \left(\frac{\partial f}{\partial y}\right)_{P_0} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Def The **gradient vector** (or, just gradient) of $f(x,y)$ at point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \left.\left(\frac{\partial f}{\partial x}\right)\right|_{P_0}\hat{i} + \left.\left(\frac{\partial f}{\partial y}\right)\right|_{P_0}\hat{j},$$

where the partial derivatives are evaluated at $P_0(x_0, y_0)$.

"gradient of f"
"del f", "grad f"

Hence the directional derivative of f along $\hat{u}$ (unit vector) at $P_0(x_0, y_0)$ is

Scalar product

$$\left(D_{\hat{u}} f\right)_{P_0} = \left(\nabla f\right)_{P_0} \cdot \hat{u}$$

Prob 2 $\quad f(x,y) = \ln(x^2+y^2)$, $P_0(1,1)$. Find gradient of f at P_0, Sketch the gradient and level curve passing through P_0.

$$\left(\nabla f\right)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \hat{j} = \left(\frac{2x+0}{x^2+y^2}\right)_{P_0} \hat{i} + \left(\frac{0+2y}{x^2+y^2}\right)_{P_0} \hat{j}$$

$$= \frac{2}{2}\hat{i} + \frac{2}{2}\hat{j} = \hat{i} + \hat{j}.$$

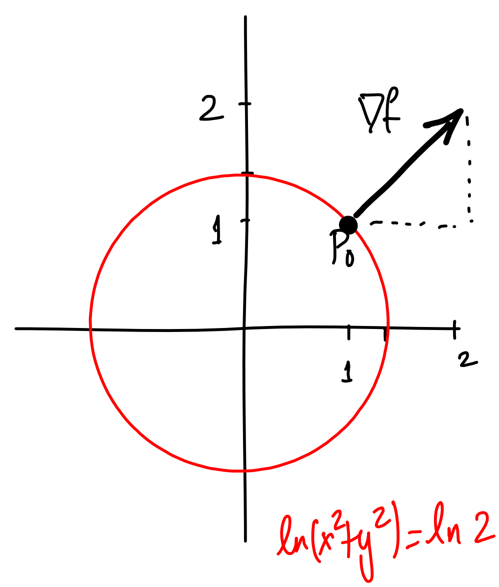
$f(x,y) = \ln(x^2+y^2)$

at $P_0(1,1)$, $f(1,1) = \ln(1^2+1^2) = \ln 2$.

The level curve is $f(x,y) = \ln 2$

$$\ln(x^2+y^2) = \ln 2$$

i.e., $\quad x^2+y^2 = 2$



Definition of ∇f naturally extends to 3 (or higher) dimensions: for $f(x,y,z)$,

$$\nabla f\Big|_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\hat{j} + \left(\frac{\partial f}{\partial z}\right)_{P_0}\hat{k}$$

<u>Prob 7</u> $f(x,y,z) = x^2+y^2-2z^2+z\ln x$, $P_0(1,1,1)$, find $(\nabla f)_{P_0}$.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(2x + \frac{z}{x}\right)\hat{i} + (2y)\hat{j} + (-4z+\ln x)\hat{k}$$ plug in $(1,1,1)$

$$= \left(2\times 1 + \frac{1}{1}\right)\hat{i} + (2\times 1)\hat{j} + (-4(1)+\ln 1)\hat{k} = 3\hat{i}+2\hat{j}-4\hat{k}.$$

Equivalently, $\nabla f = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$.

Prob 13 $g(x,y)=\dfrac{x-y}{xy+2}$, $P_0(1,-1)$. Find the directional derivative of g at P_0 in the direction of $\bar{u}=12\hat{i}+5\hat{j}$.

First, we find the unit vector in the direction of $\bar{u}$.

$$\hat{u}=\frac{\bar{u}}{\|\bar{u}\|}=\frac{12}{13}\hat{i}+\frac{5}{13}\hat{j}$$

length (or norm) of $\bar{u}=\sqrt{12^2+5^2}=13$

My notation: lower case letters with a bar denote vectors e.g., $\bar{u}$ or $\bar{a}$ ...

lower case letters with a hat ^: unit vectors. $\hat{u}$, $\hat{y}$ etc. $\|\hat{u}\|=1$.

$$\nabla g=\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}$$

$$=\frac{(xy+2)(1-0)-(x-y)(y+0)}{(xy+2)^2}\hat{i}+\frac{(xy+2)(0-1)-(x-y)(x+0)}{(xy+2)^2}\hat{j}$$

$$=\frac{(\cancel{x}y+2)-\cancel{xy}+y^2}{(xy+2)^2}\hat{i}+\frac{-(\cancel{xy}+2)-x^2+\cancel{xy}}{(xy+2)^2}\hat{j}$$ plug in $P_0(1,-1)$

$$=\frac{y^2+2}{(xy+2)^2}\hat{i}-\frac{(x^2+2)}{(xy+2)^2}\hat{j}$$

$$=\frac{3}{1}\hat{i}+\frac{-3}{1}\hat{j}=3\hat{i}-3\hat{j}$$

$$\left(D_{\hat{u}}g\right)_{P_0} = \nabla g \cdot \hat{u} = \left(3\hat{i} - 3\hat{j}\right) \cdot \left(\frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}\right)$$

$$= \frac{3 \times 12}{13} - \frac{3 \times 5}{13} = \frac{21}{13}.$$

Notice the three steps in such problems. First, we find the unit vector $\hat{u}$ in the direction of $\bar{u}$. Then we find the gradient ∇g at the given point P_0. Third, we take the scalar product of ∇g and $\hat{u}$ to get $\left(D_{\bar{u}}g\right)_{P_0}$.

MATH 273 – Lecture 8 (09/18/2014)

Gradient and Directional Derivative

$$D_{\hat{u}} f \Big|_{P_0} = \nabla f \Big|_{P_0} \cdot \hat{u} = |\nabla f| \underbrace{|\hat{u}|}_{=1\ (\hat{u} \text{ is unit vector})} \cos\theta = |\nabla f_{P_0}| \cos\theta$$

where θ is the angle between ∇f and $\hat{u}$.

Using the definition of scalar product, we can infer properties of directional derivative.

Properties of directional derivatives

① f increases most rapidly when $\theta = 0$, so that $\cos\theta = 1$, i.e., along ∇f itself. The derivative here is $|\nabla f|$.

② f decreases most rapidly when $\theta = \pi$, so that $\cos\theta = -1$. The derivative here is $-|\nabla f|$.

③ If $\hat{u}$ is orthogonal or perpendicular to ∇f, the directional derivative is zero, as $\theta = \frac{\pi}{2}$, $\cos\theta = 0$. $(D_{\hat{u}} f) = 0$.

Hence the gradient gives the direction of fastest increase (among all possible directions). Going opposite to the gradient sees the fastest decrease in the function.

Prob 21 $f(x,y,z) = \frac{x}{y} - yz$. $P_0(4,1,1)$.

Find the directions in which f increases and decreases most rapidly at P_0. Then find the derivatives in these directions.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\frac{1}{y} - 0\right)\hat{i} + \left(\frac{-x}{y^2} - z\right)\hat{j} + (0 - y)\hat{k}$$

$$= \frac{1}{y}\hat{i} - \left(\frac{x}{y^2} + z\right)\hat{j} - y\hat{k}$$

At $P_0(4,1,1)$, $\nabla f_{P_0} = \frac{1}{1}\hat{i} - \left(\frac{4}{1^2} + 1\right)\hat{j} - 1\hat{k} = \hat{i} - 5\hat{j} - \hat{k}$.

Direction of largest increase $\hat{u} = \frac{\nabla f}{\|\nabla f\|}$ (unit vector along ∇f).

i.e., $\hat{u} = \frac{1}{\sqrt{27}}\left(\hat{i} - 5\hat{j} - \hat{k}\right)$.

Notice that $\|\nabla f\| = \sqrt{(1)^2 + (-5)^2 + (-1)^2} = \sqrt{27} = 3\sqrt{3}$.

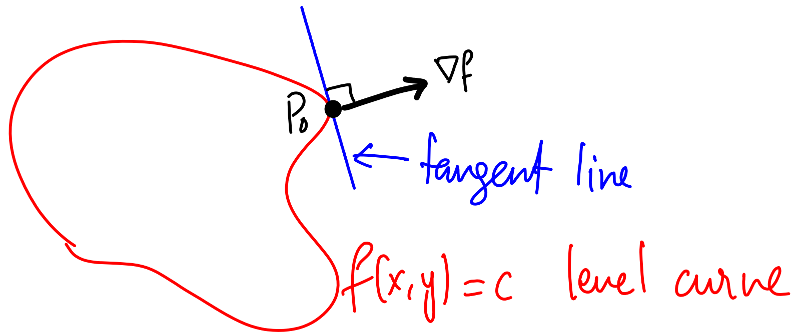
Direction of largest decrease $\hat{v} = -\hat{u} = \frac{-1}{\sqrt{27}}\left(\hat{i} - 5\hat{j} - \hat{k}\right)$.

$\left(D_{\hat{u}}f\right)_{P_0} = |\nabla f|$ (as $\left(D_{\hat{u}}f\right) = |\nabla f|\cos 0$) $= \sqrt{27}$. ← largest derivative in any direction

Similarly, $\left(D_{\hat{v}}f\right)_{P_0} = -|\nabla f| = -\sqrt{27}$. ← largest decrease, i.e., most negative derivative

Gradients and Tangents Level Curves

Intuition

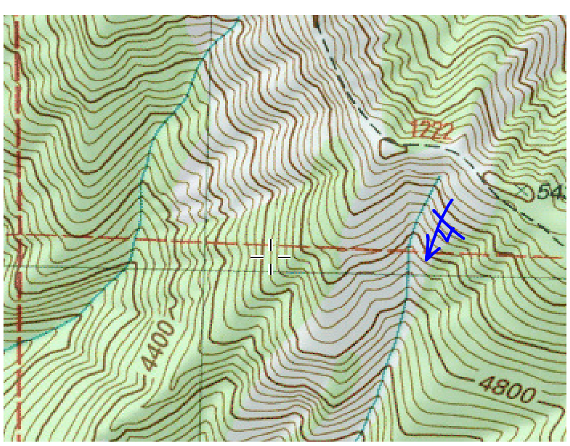


∇f is always normal (or orthogonal) to the level curves.

The tangent at any point on $f(x,y)=c$ gives the direction in which the function is not changing at that point (ie., instantaneously). Since the gradient is perpendicular to the direction in which the derivative is zero, it is perpendicular to the tangent. By finding the gradient, we can find the equation to the tangent line.

We could also prove this result formally – see the textbook.

Illustration: rivers on topo maps are always perpendicular to the contour lines, which are level curves of elevation.



(from gmaps)

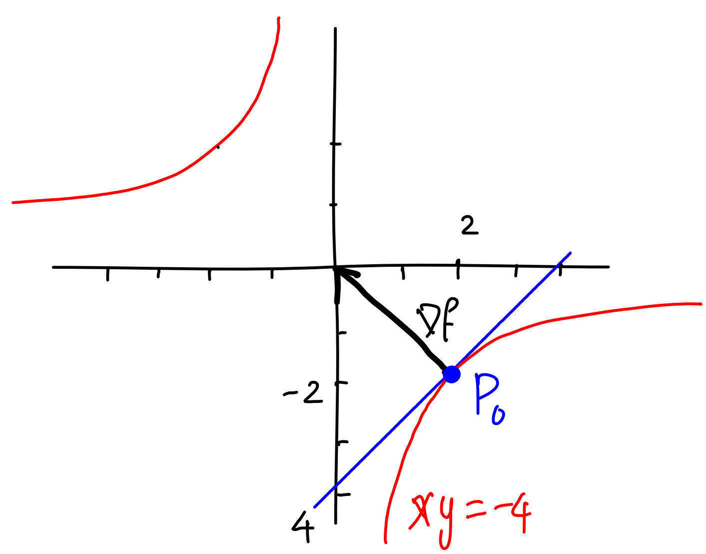
On topographical maps, the contour lines are level curves of elevation. Rivers and streams, indicated by blue lines/curves always flow perpendicular to the contour lines, as water seeks the most direct path to lower elevation.

We can write down the equation of the tangent line at point P_0 on the level curve $f(x,y)=c$ by finding the gradient of f at P_0. The slope of the tangent line is $-\dfrac{1}{\text{slope of } (\nabla f)_{P_0}}$. $\longrightarrow$ Recall that the slope of the line perpendicular to a given line with slope m_1 is $-\frac{1}{m_1}$.

<u>Prob 27</u> $\underbrace{xy=-4}_{f(x,y)=c}$, $P_0(2,-2)$. Sketch the level curve $f(x,y)=c$ and ∇f at P_0. Then write the equation for the tangent to the curve at P_0.

$$xy=-4$$

i.e., $y = \dfrac{-4}{x}$



$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= y\hat{i} + x\hat{j}$$

at $P_0(2,-2)$, $(\nabla f)_{P_0} = -2\hat{i} + 2\hat{j}$.

Slope of the tangent line at P_0 will be $\dfrac{-1}{\text{Slope of } \nabla f}$.

Slope of $\nabla f = \dfrac{2}{-2} = -1$. Hence, slope of tangent line $= \dfrac{-1}{-1} = 1$.

Hence the tangent line, which passes through $P_0(2,-2)$, is

$$(y - -2) = 1(x-2), \quad \text{i.e,} \quad y = x-4.$$

The equation of the line with slope m and passing through the point (x_0, y_0) is $(y-y_0) = m(x-x_0)$.

Prob 29(d) $f(x,y) = x^2 - xy + y^2 - y$. Find direction $\hat{u}$ along which $D_{\hat{u}} f(1,-1) = 4$.

This is a "reverse" problem – you know how to find the directional derivative of the function along a given direction. Here, you are given the directional derivative and are asked to find the direction.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = (2x - y + 0 - 0)\hat{i} + (0 - x + 2y - 1)\hat{j}$$

$$(\nabla f)_{(1,-1)} = 3\hat{i} - 4\hat{j}$$

Let $\hat{u} = u_1 \hat{i} + u_2 \hat{j}$. $\|\hat{u}\| = 1$ so $u_1^2 + u_2^2 = 1$ ——— (1).

We specify the direction as a unit vector. We start with this unknown vector as $\hat{u} = u_1 \hat{i} + u_2 \hat{j}$, and solve for the two unknowns u_1 and u_2. We need two equations.

Sine $(D_{\hat{u}} f)_{(1,-1)} = 4$, we get

$$(\nabla f)_{(1,-1)} \cdot \hat{u} = 4$$

$$(3\hat{i} - 4\hat{j}) \cdot (u_1 \hat{i} + u_2 \hat{j}) = 4$$

$$3u_1 - 4u_2 = 4 \quad \text{——— (2)}$$

(2) gives $3u_1 = 4u_2 + 4$, So $u_1 = \frac{4}{3}(u_2 + 1)$.

Plugging into (1) gives

$$\left(\frac{4}{3}\right)^2 (u_2 + 1)^2 + u_2^2 = 1$$

$$16(u_2^2 + 2u_2 + 1) + 9u_2^2 = 9$$

$$25u_2^2 + 32u_2 + 7 = 0$$

notice that $25 + 7 = 32$, which points to the factorization desired

$$(25u_2 + 7)(u_2 + 1) = 0$$

i.e., $u_2 = -\frac{7}{25}$, $u_2 = -1$.

The corresponding values of u_1 are (given by $u_1 = \frac{4}{3}(u_2+1)$)

$$u_1 = \frac{4}{3}\left(\frac{-7}{25}+1\right) \quad \text{and} \quad u_1 = \frac{4}{3}(-1+1).$$

Hence $u_1 = \frac{4}{3}\left(\frac{18}{25}\right) = \frac{24}{25}$ and $u_1 = 0$.

Thus, there are two directions $\hat{u}$ along which $(D_{\hat{u}}f)_{P_0} = 4$. They are $\hat{u} = 0\hat{i} + (-1)\hat{j}$, i.e., $\hat{u} = -\hat{j}$, and

$$\hat{u} = \frac{24}{25}\hat{i} - \frac{7}{25}\hat{j}.$$

MATH 273 – Lecture 9 (09/23/2014)

Prob 35 (Section 13.5)

$$\left(D_{\hat{u}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \hat{u}$$

∇f → gradient

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Setting: You're given $\left(D_{\bar{u}}f\right)_{P_0}$ and $\left(D_{\bar{v}}f\right)_{P_0}$, and asked to find $\left(D_{\bar{w}}f\right)_{P_0}$ for directions $\bar{u}$, $\bar{v}$, and $\bar{w}$ (not necessarily unit vectors). Form of f is not given.

$(D_{\bar{u}}f)$ at $P_0(1,2)$ (x, y) in direction of $\bar{u} = \hat{i} + \hat{j}$ is $2\sqrt{2}$ and in direction of $\bar{v} = -2\hat{j}$ is -3. Find $(D_{\bar{w}}f)_{P_0}$ in the direction of $\bar{w} = -\hat{i} - 2\hat{j}$.

$$\left(D_{\bar{w}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \hat{w} = (\nabla f)_{P_0} \cdot \frac{\bar{w}}{\|\bar{w}\|}$$

Let $(\nabla f)_{P_0} = (f_x)_{P_0}\hat{i} + (f_y)_{P_0}\hat{j} = f_1\hat{i} + f_2\hat{j}$ where f_1 and f_2 are unknown.

$$\left(D_{\bar{u}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \frac{\bar{u}}{\|\bar{u}\|} = (f_1\hat{i} + f_2\hat{j}) \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}} = 2\sqrt{2}$$

i.e., $\sqrt{2}\left(\frac{f_1}{\sqrt{2}} + \frac{f_2}{\sqrt{2}} = 2\sqrt{2}\right)$, which gives $f_1 + f_2 = 4$ —(1)

$$\left(D_{\bar{v}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \frac{\bar{v}}{\|\bar{v}\|} = (f_1\hat{i} + f_2\hat{j}) \cdot \frac{-2\hat{j}}{2} = \underbrace{\frac{f_2 \cdot -2}{2}}_{\left(f_1 \times 0 + f_2 \times \frac{-2}{2}\right)} = -3$$

i.e., $f_2 = 3$ ——— (2). So, (1) gives $f_1 = 1$.

$$(\nabla f)_{P_0} = 1\hat{i} + 3\hat{j} = \hat{i} + 3\hat{j}.$$

Hence $\left(D_{\bar{w}} f\right)_{P_0} = (\nabla f)_{P_0} \cdot \dfrac{\bar{w}}{\|\bar{w}\|}$

$\bar{w} = -\hat{i} - 2\hat{j}$

$\|\bar{w}\| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$

$$= (\hat{i} + 3\hat{j}) \cdot \frac{(-\hat{i} - 2\hat{j})}{\sqrt{5}} = \left(1 \times \frac{-1}{\sqrt{5}} + 3 \times \frac{-2}{\sqrt{5}}\right) = \frac{-7}{\sqrt{5}}.$$

Tangent Planes and Differentials (Section 13.6)

Extend idea of tangent line of a level curve to that of the tangent plane of a level surface.

Let $\bar{r}(t) = \underbrace{x(t)}_{x}\,\hat{i} + \underbrace{y(t)}_{y}\,\hat{j} + \underbrace{z(t)}_{z}\,\hat{k}$ be a smooth curve on a level surface $f(x, y, z) = c$.

We write $f(x(t), y(t), z(t)) = c$; and apply chain rule (w.r.t t).

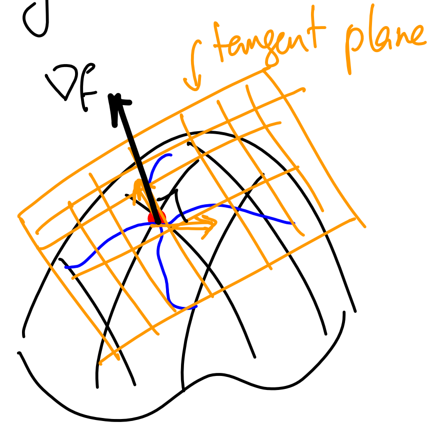
$$\frac{df}{dt} = \frac{dc}{dt} = 0$$

$$\underbrace{\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)}_{\frac{d\bar{r}}{dt} \longrightarrow \text{velocity vector}} = 0$$

Hence ∇f is orthogonal to the velocity vector.

All tangent lines to the surface (i.e., tangents to all level curves on the surface) at given point P_0 lie on a plane that is orthogonal to ∇f at P_0.



Def The **tangent plane** at P_0 on the level surface $f(x,y,z)=c$ of a differentiable function f is the plane through P_0 orthogonal to $(\nabla f)_{P_0}$.

The line parallel to $(\nabla f)_{P_0}$ passing through P_0 is the **normal line** of the surface at P_0.

Section 11.5 : Equation of plane perpendicular to $A\hat{i}+B\hat{j}+C\hat{k}$ at $P_0(x_0,y_0,z_0)$ is $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.

Equation of a line through P_0 parallel to $\bar{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ is $\bar{r}(t)=\bar{r}_0+t\bar{v}$ for $-\infty<t<\infty$, i.e.,

$$\bar{r}(t)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix}+t\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix},$$

i.e.,

$$x=x_0+tv_1$$
$$y=y_0+tv_2$$
$$z=z_0+tv_3$$

Hence, the tangent plane to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$ is

$$\left(\frac{\partial f}{\partial x}\right)_{P_0}(x-x_0)+\left(\frac{\partial f}{\partial y}\right)_{P_0}(y-y_0)+\left(\frac{\partial f}{\partial z}\right)_{P_0}(z-z_0)=0.$$

The normal line is given by

$$x=x_0+\left(\frac{\partial f}{\partial x}\right)_{P_0}t,\quad y=y_0+\left(\frac{\partial f}{\partial y}\right)_{P_0}t,\quad z=z_0+\left(\frac{\partial f}{\partial z}\right)_{P_0}t,\quad -\infty<t<\infty.$$

<u>Prob 2</u> Find equations of (a) tangent plane and (b) normal line at $P_0(3,5,-4)$ to the surface $\underbrace{x^2+y^2-z^2=18}_{f(x,y,z)=c}$.

$$\nabla f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$$
$$=2x\hat{i}+2y\hat{j}-2z\hat{k}$$

at $P_0(\underset{x_0,y_0,z_0}{3,5,-4})$, $\nabla f=6\hat{i}+10\hat{j}+8\hat{k}$.

So, the tangent plane is $6(x-3)+10(y-5)+8(z+4)=0$

i.e., $\frac{1}{2}\left(6x+10y+8z+(-18-50+32)=0\right)$

i.e., $3x+5y+4z=18$.

Normal line is $x=3+6t$,

$y=5+10t$

$z=-4+8t$, $\quad -\infty<t<\infty$.

MATH 273 – Lecture 10 (09/25/2014)

Sections covered up to end of today's lecture will be relevant for Exam 1, on Thursday Oct 2.

Hw 5 is due next Tuesday by 5:00 pm.

Tangent plane and normal line to surface $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$

Tangent plane: $(f_x)_{P_0}(x-x_0)+(f_y)_{P_0}(y-y_0)+(f_z)_{P_0}(z-z_0)=0.$

normal line: $x=x_0+(f_x)_{P_0}t,\ y=y_0+(f_y)_{P_0}t,\ z=z_0+(f_z)_{P_0}t,\ -\infty<t<\infty.$

<u>Prob 5</u> Find tangent plane and normal line at $P_0(0,1,2)$ to surface $\underbrace{\cos\pi x - x^2y + e^{xz} + yz = 4}_{f(x,y,z)=c}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath}+\frac{\partial f}{\partial y}\hat{\jmath}+\frac{\partial f}{\partial z}\hat{k} = \left(-\pi\sin(\pi x)-2xy+ze^{xz}+0\right)\hat{\imath}$$
$$+\left(0-x^2+0+z\right)\hat{\jmath}+\left(0-0+xe^{xz}+y\right)\hat{k}$$

At $P_0(\underset{x_0,y_0,z_0}{0,1,2})$, $$(\nabla f)_{P_0}=(0-0+2e^0)\hat{\imath}+(-0^2+2)\hat{\jmath}+(0\cdot e^0+1)\hat{k}$$
$$=2\hat{\imath}+2\hat{\jmath}+\hat{k}.$$

So, tangent plane is $2(x-\underset{x_0}{0})+2(y-\underset{y_0}{1})+1(z-\underset{z_0}{2})=0$

i.e, $2x+2y+z-4=0$

Normal line is given by: $x=0+2t,\ y=1+2t,\ z=2+1t,\ -\infty<t<\infty,$

i.e., $x=2t,\ y=1+2t,\ z=2+t.$

We could also specify the surface in the form $z=f(x,y)$, instead of $f(x,y,z)=c$.

<u>Prob 9</u> $z=\ln(x^2+y^2)$, $P_0(1,0,0)$. Find the tangent plane.

$z=\ln(x^2+y^2)$ can be written as $\underbrace{\ln(x^2+y^2)-z=0}_{f(x,y,z)=c}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k} = \frac{2x}{(x^2+y^2)}\hat{i}+\frac{2y}{(x^2+y^2)}\hat{j}-1\hat{k}$$

$$(\nabla f)_{P_0} = \frac{2}{1^2}\hat{i}+\frac{0}{1^2}\hat{j}-\hat{k} = 2\hat{i}-\hat{k}.$$

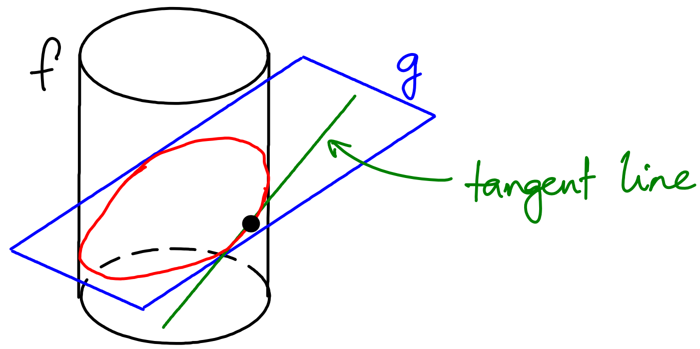
$(x_0,y_0,z_0)=(1,0,0)$

So the tangent plane is $2(x-1)+0(y-0)+(-1)(z-0)=0$

i.e., $2x-z-2=0$

Intersection of two surfaces could generate a curve of intersection, and we could use the gradients to both surfaces at a point on this curve to find the line tangent to both surfaces at that point.

Say the two surfaces are $f(x,y,z)=c_1$ and $g(x,y,z)=c_2$. The tangent line in question is perpendicular to both $(\nabla f)_{P_0}$ and $(\nabla g)_{P_0}$.



Prob 15 $\underbrace{x^2+2y+2z=4}_{z=2-(\frac{1}{2})x^2-y}$, $y=1$, $P_0(1,1,\frac{1}{2})$

$f(x,y,z)=c_1$ $\qquad$ $g(x,y,z)=c_2$

$\nabla f = 2x\,\hat{i} + 2\hat{j} + 2\hat{k}$ $\qquad$ $\nabla g = 0\hat{i}+1\hat{j}+0\hat{k} = \hat{j}$

$(\nabla f)_{P_0} = 2\hat{i} + \overset{2\cdot 1}{2}\hat{j} + \overset{2\times 1}{2}\hat{k}$ $\qquad$ $(\nabla g)_{P_0} = \hat{j}$

We need to find a direction orthogonal to both ∇f and ∇g.

$$\bar{v} = \nabla f \times \nabla g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = (2\times 0 - 1\times 2)\hat{i} - (2\times 0 - 0\times 2)\hat{j} + (2\times 1 - 0\times 2)\hat{k}$$

determinant

$P_0(\overset{x_0}{1},\overset{y_0}{1},\overset{z_0}{\tfrac{1}{2}})$ $\qquad$ $= -2\hat{i}+2\hat{k}$

The tangent line is $x = x_0 - 2t$, $y = y_0 + 0t$, $z = z_0 + 2t$

i.e., $x = 1-2t$, $y=1$, $z=\frac{1}{2}+2t$, $-\infty < t < \infty$.

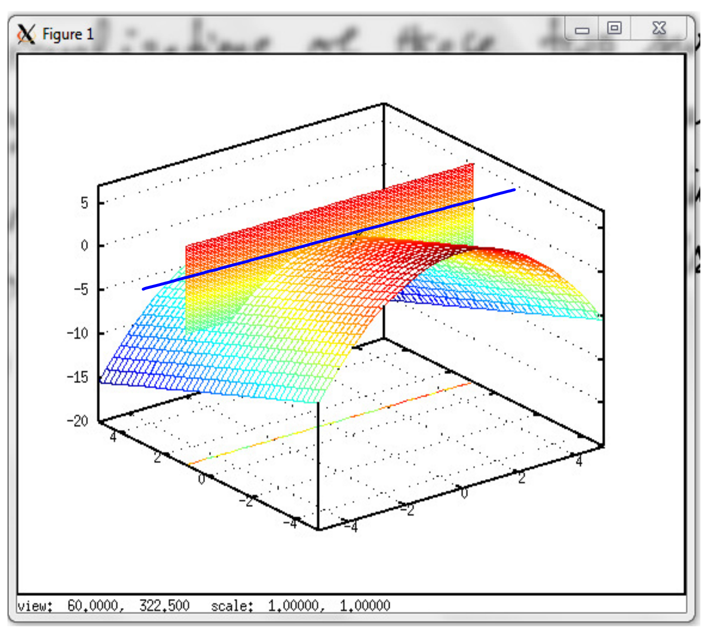
We will visualize these two surfaces. While it is not necessary to use a computer to create such visualizations, it could be very instructive.

We could produce visualizations of these two surfaces in Octave/Matlab. It is simpler to create visualizations when the surface is given as $z = f(x,y)$, but certain packages such as Mathematica and Gnuplot could also plot implicitly defined surfaces directly.

```
% Problem 15 from Section 13.6
% x^2 + 2y + 2z = 4 and y=1 at P0 (1,1,1/2)

tx = ty = linspace (-5,5,41)';
[xx,yy] = meshgrid (tx,ty);
tz = 2 - (1/2)*xx.^2 - yy;     % z = 2 - (1/2)x^2 - y
mesh(tx,ty,tz);
hold on;

meshc(xx,ones(size(xx)),yy);   % y = 1

plot3(1,1,1/2,"x","markersize",10,"color","k")
```

→ These are the commands in Octave to plot the two surfaces. We generate a regular grid of points (x,y) using meshgrid, and evaluate the function at each such point.

This is one view of the visualization generated. One could rotate the view in Octave. The tangent line is drawn on top here by hand (but one could do it in Octave as well).



Prob 17 $\underbrace{x^3+3x^2y^2+y^3+4xy-z^2}_{f}=0$

Find the equation to the line tangent to both surfaces at P_0.

$\underbrace{x^2+y^2+z^2}_{g}=11, \quad P_0(1,1,3)$

$$\nabla f = (3x^2+6xy^2+4y)\hat{i} + (6x^2y+3y^2+4x)\hat{j} + (-2z)\hat{k}$$

$$\nabla g = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$(\nabla f)_{(1,1,3)} = 13\hat{i}+13\hat{j}-6\hat{k}, \quad (\nabla g)_{(1,1,3)} = 2\hat{i}+2\hat{j}+6\hat{k}$$

$$\bar{v} = \begin{vmatrix} i & j & k \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = 90\hat{i}-90\hat{j}+0\hat{k} = 90\hat{i}-90\hat{j}$$

$$(x_0,y_0,z_0)=(1,1,3)$$

Tangent line: $x = 1+90t$
$y = 1-90t \qquad -\infty<t<\infty$
$z = 3$

MATH 273 – Lecture 11 (09/30/2014)

Estimating changes in a specific direction

In 1D, change in $f(x)$ at $x = p_0$ is estimated by

$$\underbrace{df}_{\text{differential of } f \text{ at } x=p_0} = \underbrace{f'(p_0)\cdot dx}_{\text{derivative} \times \text{increment}} \quad \text{for small increment } dx$$

Extending to higher dimensions,

$$df = \underbrace{\left(\left(\nabla f\right)_{P_0} \cdot \hat{u}\right) ds}_{\text{directional derivative} \times \text{increment in the direction of } \hat{u}},$$ where ds is the change in the direction of $\hat{u}$.

Prob 21 By about how much will $g(x,y,z) = x + x\cos z - y\sin z + y$ change when $P(x,y,z)$ moves from $P_0(2,-1,0)$ toward the point $P_1(0,1,2)$ a distance of $ds = 0.2$ units?

$$\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$$

$$= (1 + \cos z - 0 + 0)\hat{i} + (0 + 0 - \sin z + 1)\hat{j} + (0 - x\sin z - y\cos z + 0)\hat{k}$$

$$= (1 + \cos z)\hat{i} + (1 - \sin z)\hat{j} - (x\sin z + y\cos z)\hat{k}$$

$$(\nabla g)_{P_0} = (1 + \underbrace{\cos 0}_{1})\hat{i} + (1 - \underbrace{\sin 0}_{0})\hat{j} - (2\underbrace{\sin 0}_{0} + (-1)\underbrace{\cos 0}_{1})\hat{k}$$

$$= 2\hat{i} + \hat{j} + \hat{k}.$$

Direction $\bar{u} = \overrightarrow{P_0P_1} = (0-2)\hat{i} + (1-1)\hat{j} + (2-0)\hat{k}$

$P_0(2,-1,0)$ $\qquad = -2\hat{i} + 2\hat{j} + 2\hat{k}$

$P_1(0,1,2)$ $\qquad \|u\| = \sqrt{(-2)^2+(2)^2+(2)^2} = 2\sqrt{3}.$

So, $\hat{u} = \dfrac{\bar{u}}{\|\bar{u}\|} = \dfrac{1}{2\sqrt{3}}\left(-2\hat{i}+2\hat{j}+2\hat{k}\right) = \dfrac{1}{\sqrt{3}}\left(-\hat{i}+\hat{j}+\hat{k}\right).$

$$\left(D_{\hat{u}}g\right)_{P_0} = (\nabla g)_{P_0} \cdot \hat{u} = \left(2\hat{i}+\hat{j}+\hat{k}\right)\cdot \frac{1}{\sqrt{3}}\left(-\hat{i}+\hat{j}+\hat{k}\right)$$

$$= \frac{1}{\sqrt{3}}\left(2\times -1 + 1\times 1 + 1\times 1\right) = 0.$$

So $dg = \left(D_{\hat{u}}g\right)_{P_0} \cdot ds = 0.(0.2) = 0.$

Review for Exam 1

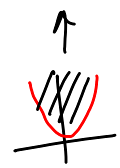
Exam 1 covers 13.1, 13.3, 13.4, and 13.5.

- 13.1 → domain, range, level curves, open/closed, bounded/unbounded
- 13.3 → partial derivatives
- 13.4 → chain rule, branch diagrams
- 13.5 → gradient & directional derivative

Practice Exam

8. True/False

(a) False. Take $y \geq x^2$ is closed, as it includes its boundary $y = x^2$. But it is unbounded



(b) False. We draw two branch diagrams, one for each independent variable.

(c) True. Follows from properties of ∇f.

(d) False. $(D_{\hat{u}} f) = \nabla f \cdot \hat{u} = |\nabla f| \underbrace{|\hat{u}|}_{=1} \cos\theta$

If $|\nabla f| = 0$, then $(D_{\hat{u}} f) = 0$ in all directions $\hat{u}$.

2(a). $f(x,y) = \dfrac{x+y}{xy-1}$

$$\frac{\partial f}{\partial x} = \frac{(xy-1)(1+0)-(x+y)(y-0)}{(xy-1)^2} = \frac{\cancel{xy}-1-\cancel{xy}-y^2}{(xy-1)^2} = -\frac{(y^2+1)}{(xy-1)^2}$$

$f(x,y)$ is symmetric w.r.t x and y, so

$$\frac{\partial f}{\partial y} = \frac{-(x^2+1)}{(yx-1)^2} = -\frac{(x^2+1)}{(xy-1)^2}.$$

$$f(y,x) = \frac{y+x}{yx-1} = \frac{x+y}{xy-1} = f(x,y)$$

Alternatively, we could evaluate $\dfrac{\partial f}{\partial y}$ directly:

$$\frac{\partial f}{\partial y} = \frac{(xy-1)(0+1)-(x+y)(x-0)}{(xy-1)^2} = \frac{(\cancel{xy}-1)-x^2-\cancel{xy}}{(xy-1)^2}$$

$$= \frac{-(x^2+1)}{(xy-1)^2}.$$

MATH 273 – Lecture 13 (10/07/2014)

Average on Exam 1: ~55.5

Offer: * If you score 90+ in Exam 2, your Exam 2 score will replace your Exam 1 score.

* If you score 85-89 in Exam 2, the weights for Exams 1 and 2 will be 5% and 35%, respectively
* If you score 80-84 in Exam 2, the weights for Exams 1 and 2 will be 10% and 30%, respectively.

Offer also applies with Exam 1 and 2 swapped!

Need to show up for the Exams to apply!!

Hw 6 due this FRIDAY, Oct 10, by 4 PM in my mailbox (VSCI 130).

Linearization of $f(x,y)$

In 1D, for a function $f(x)$ that is differentiable at $x=a$,

$L(x) = f(a) + f'(a)(x-a)$ is the linearization (or linear approximation) of $f(x)$ at $x=a$.

$f'(a)dx$ is the differential of f at $x=a$

estimates the small change in f for a small change dx in x.

Extending to 2- and higher dimensions, at $P_0(x_0,y_0)$

$$L(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{P_0}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{P_0}(y-y_0)$$ is

the **standard linear approximation** of f at P_0.

total differential of f at P_0

$$= f_x\big|_{P_0} dx + f_y\big|_{P_0} dy + \underbrace{f_z\, dz}_{\text{if we have 3 variables}}$$

The linearization is a good approximation to $f(x,y)$ only for small changes in x and y, i.e., when $x-x_0$ and $y-y_0$ (or dx, dy) are small.

13-3

Prob 29 $f(x,y) = e^x \cos y$. Find standard linear approximation $L(x,y)$ of $f(x,y)$ at (a) $(0,0)$ and (b) $(0, \pi/2)$.

$$L(x,y)\Big|_{P_0} = f(x_0,y_0) + f_x\Big|_{P_0}(x-x_0) + f_y\Big|_{P_0}(y-y_0)$$

$$f_x = e^x \cos y \qquad f_y = -e^x \sin y$$

(a) $P_0\,(\underset{x_0,\,y_0}{0,0})$

$$f(x_0,y_0) = e^0 \cos 0 = 1$$

$$f_x = e^0 \cos 0 = 1$$

$$f_y = -e^0 \sin 0 = 0$$

$$L(x,y) = \underbrace{1}_{f(x_0,y_0)} + \underbrace{1}_{f_x}(x - \underbrace{0}_{x_0}) + \overbrace{0}^{f_y}(y - \overbrace{0}^{y_0})$$

$$= 1 + x.$$

(b) $P_0\,(\underset{x_0,\,y_0}{0, \pi/2})$

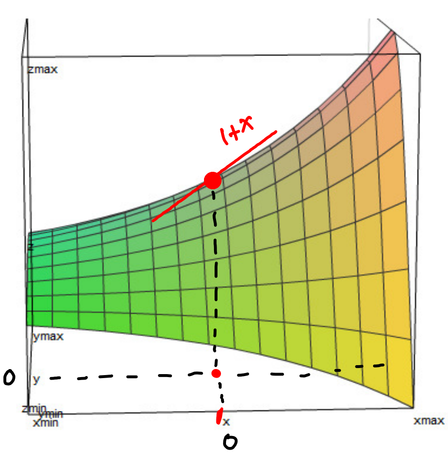
$$f(x_0,y_0) = e^0 \cos \pi/2 = 1 \times 0 = 0$$

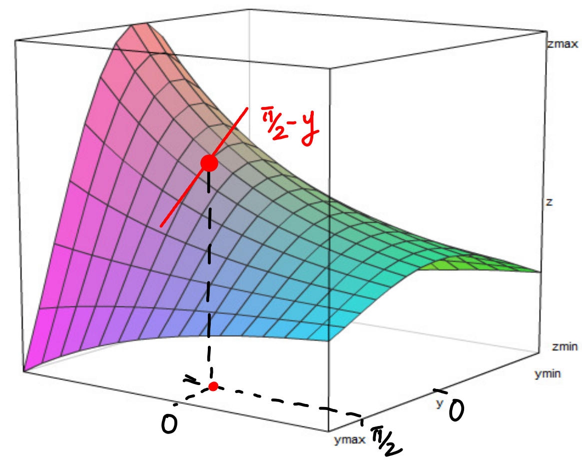
$$f_x = e^0 \cos \pi/2 = 0$$

$$f_y = -e^0 \sin \pi/2 = -1$$

$$L(x,y) = \underbrace{0}_{f(x_0,y_0)} + \underbrace{0}_{f_x}(x - \underbrace{0}_{x_0}) + \underbrace{(-1)}_{f_y}(y - \underbrace{\pi/2}_{y_0})$$

$$= \pi/2 - y$$





(13-4)

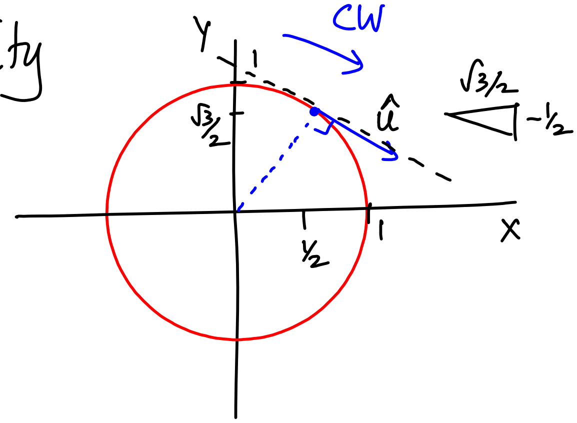
Prob 23 Temperature $T(x,y) = x \sin 2y$. A particle is moving clockwise along unit circle (centered at origin) at speed 2 m/s.

(a) How fast is T changing at $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ per meter?

(b) How fast is T changing with time at P?

$\hat{u}$: unit vector along the velocity direction

$$\hat{u} = \frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}$$



(a) $$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j}$$

$$= \sin 2y\,\hat{i} + 2x\cos 2y\,\hat{j}$$

$$(\nabla T)_P = \sin\sqrt{3}\,\hat{i} + 2\left(\tfrac{1}{2}\right)\cos\sqrt{3}\,\hat{j} = \sin\sqrt{3}\,\hat{i} + \cos\sqrt{3}\,\hat{j}$$

$P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$(D_{\hat{u}}T)_P = \nabla T_P \cdot \hat{u} = \left(\sin\sqrt{3}\,\hat{i} + \cos\sqrt{3}\,\hat{j}\right)\cdot\left(\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}\right)$$

$$= \frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3} = 0.935\ {}^\circ C/m$$

(b) $$\frac{dT}{dt} = \nabla T \cdot \left(\frac{d\bar{r}}{dt}\right) \longrightarrow \text{velocity vector } \bar{v}$$

$$\frac{d\bar{r}}{dt} = |\bar{v}|\cdot\hat{u} = 2\left(\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}\right)$$

$$= \left(\sin\sqrt{3}\,\hat{i} + \cos\sqrt{3}\,\hat{j}\right)\cdot 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\hat{j}\right) = 2 \times 0.935 = 1.87\ {}^\circ C/s.$$

31. Wind chill factor Wind chill, a measure of the apparent temperature felt on exposed skin, is a function of air temperature and wind speed. The precise formula, updated by the National Weather Service in 2001 and based on modern heat transfer theory, a human face model, and skin tissue resistance, is

$$W = W(v, T) = 35.74 + 0.6215\,T - 35.75\,v^{0.16} + 0.4275\,T \cdot v^{0.16},$$

where T is air temperature in °F and v is wind speed in mph. A partial wind chill chart is given.

T(°F)

v (mph)	**30**	**25**	**20**	**15**	**10**	**5**	**0**	**−5**	**−10**
5	25	19	13	7	1	−5	−11	−16	−22
10	21	15	9	3	−4	−10	−16	−22	−28
15	19	13	6	0	−7	−13	−19	−26	−32
20	17	11	4	−2	−9	−15	−22	−29	−35
25	16	9	3	−4	−11	−17	−24	−31	−37
30	15	8	1	−5	−12	−19	−26	−33	−39
35	14	7	0	−7	−14	−21	−27	−34	−41

a. Use the table to find $W(20, 25)$, $W(30, -10)$, and $W(15, 15)$.

b. Use the formula to find $W(10, -40)$, $W(50, -40)$, and $W(60, 30)$.

c. Find the linearization $L(v, T)$ of the function $W(v, T)$ at the point (25, 5).

d. Use $L(v, T)$ in part (c) to estimate the following wind chill values.

i) $W(24, 6)$ **ii)** $W(27, 2)$

iii) $W(5, -10)$ (Explain why this value is much different from the value found in the table.)

at $P_0(v_0, T_0)$

$$L(v,T) = W(v_0, T_0) + \left.\frac{\partial W}{\partial v}\right|_{P_0}(v - v_0) + \left.\frac{\partial W}{\partial T}\right|_{P_0}(T - T_0)$$

At $P_0(\underset{v_0}{25}, \underset{T_0}{5})$

$$\frac{\partial W}{\partial v} = -35.75 \times 0.16\, v^{(0.16-1)} + (0.4275)(T)\, 0.16\, v^{(0.16-1)}$$

$$\frac{\partial W}{\partial T} = 0.6215 + 0.4275 \cdot v^{0.16}$$

We will finish this problem in the next lecture...

MATH 273 – Lecture 14 (10/09/2014)

Wind chill factor problem – continued...

31. Wind chill factor Wind chill, a measure of the apparent temperature felt on exposed skin, is a function of air temperature and wind speed. The precise formula, updated by the National Weather Service in 2001 and based on modern heat transfer theory, a human face model, and skin tissue resistance, is

$$W = W(v, T) = 35.74 + 0.6215\,T - 35.75\,v^{0.16} + 0.4275\,T \cdot v^{0.16},$$

where T is air temperature in °F and v is wind speed in mph. A partial wind chill chart is given.

v (mph) \ T (°F)	30	25	20	15	10	5	0	−5	−10
5	25	19	13	7	1	−5	−11	−16	−22
10	21	15	9	3	−4	−10	−16	−22	−28
15	19	13	6	0	−7	−13	−19	−26	−32
20	17	11	4	−2	−9	−15	−22	−29	−35
25	16	9	3	−4	−11	−17	−24	−31	−37
30	15	8	1	−5	−12	−19	−26	−33	−39
35	14	7	0	−7	−14	−21	−27	−34	−41

a. Use the table to find $W(20, 25)$, $W(30, -10)$, and $W(15, 15)$.

b. Use the formula to find $W(10, -40)$, $W(50, -40)$, and $W(60, 30)$.

c. Find the linearization $L(v, T)$ of the function $W(v, T)$ at the point $(25, 5)$.

d. Use $L(v, T)$ in part (c) to estimate the following wind chill values.

i) $W(24, 6)$ ii) $W(27, 2)$

iii) $W(5, -10)$ (Explain why this value is much different from the value found in the table.)

at $P_0(v_0, T_0) = (\underset{v_0}{25}, \underset{T_0}{5})$

$$L(v,T) = W(v_0,T_0) + \left.\frac{\partial W}{\partial v}\right|_{P_0}(v - v_0) + \left.\frac{\partial W}{\partial T}\right|_{P_0}(T - T_0)$$

$$\frac{\partial W}{\partial v} = 0 + 0 - (35.75)(0.16)v^{(0.16-1)} + 0.4275 \cdot T \cdot (0.16)\, v^{(0.16-1)}$$
$$= -(35.75)(0.16)v^{-0.84} + (0.4275)(0.16)\,T\,v^{-0.84}$$

$$\frac{\partial W}{\partial T} = 0 + 0.6215 - 0 + 0.4275 \cdot v^{0.16}$$
$$= 0.6215 + 0.4275 \cdot v^{0.16}$$

At $P_0(25,5)$, we get $W(v_0,T_0) = -17.41$, $W_v\big|_{P_0} = -0.36$, and $W_T\big|_{P_0} = 1.34$. Hence

$$L(v,T)\Big|_{P_0} = -17.41 - 0.36(v - \overset{v_0}{25}) + 1.34(T - \overset{T_0}{5})$$
$$= -15.09 - 0.36v + 1.34T.$$

(d) $W(24,6) \approx L(24,6) = -15.71$, which is very close to $W(24,6)$ itself!

approximately

$L(27,2) = -22.14$, while $W(27,2) = -22.143$.

But, $L(5,-10) = -30.26$, while $W(5,-10) = -22.26$!

The values are very different because $(5,-10)$ is not near $P_0(25,5)$. The linearization is valid (or accurate) only close to the point at which the linearization is taken.

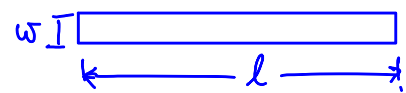
Here are the commands used in Octave (same as MATLAB):

```
octave:2> v0=25;T0=5;
octave:3> W = 35.74 + 0.6215*T - 35.75*v^(0.16) + 0.4275*T*v^(0.16);
error: `T' undefined near line 3 column 20
octave:3> v=v0; T=T0; W = 35.74 + 0.6215*T - 35.75*v^(0.16) + 0.4275*T*v^(0.16)
W = -17.409
octave:4> W_v = -(35.75)*(0.16)*v^(0.16-1) + 0.4275*0.16*T*v^(0.16-1)
W_v = -0.36004
octave:5> W_T = 0.6215 + 0.4275*v^(0.16)
W_T =  1.3370
octave:6> -17.409 - 0.36004*(-25) + 1.337*(-5)
ans = -15.093
octave:7> L = -15.093 - 0.36004*v + 1.337*T
L = -17.409

octave:8> v=24;T=6;
octave:9> L = -15.093 - 0.36004*v + 1.337*T
L = -15.712
octave:10>  W = 35.74 + 0.6215*T - 35.75*v^(0.16) + 0.4275*T*v^(0.16)
W = -15.710

octave:11> v=27;T=2;
octave:12> L = -15.093 - 0.36004*v + 1.337*T
L = -22.140
octave:13>  W = 35.74 + 0.6215*T - 35.75*v^(0.16) + 0.4275*T*v^(0.16)
W = -22.143

octave:14> v=5;T=-10;
octave:15> L = -15.093 - 0.36004*v + 1.337*T
L = -30.263
octave:16>  W = 35.74 + 0.6215*T - 35.75*v^(0.16) + 0.4275*T*v^(0.16)
W = -22.256

octave:17> v=5;T=20;
octave:18> L = -15.093 - 0.36004*v + 1.337*T
L =  9.8468
octave:19>  W = 35.74 + 0.6215*T - 35.75*v^(0.16) + 0.4275*T*v^(0.16)
W =  12.981
```

We now do a problem using the total differential.

Prob 51 We are calculating the area of a thin long rectangle by measuring length and width. Which dimension should we measure more carefully so as to minimize error in the area computed?

Area $A = lw$ $\quad l$ = length $\quad w$ = width



The total differential $dA = A_l dl + A_w dw$.

l is much larger than w

But $A_l = w$ and $A_w = l$. So

$$dA = w\,dl + l\,dw$$

We can think of dA as the error in computing the area, and dl and dw as the errors in measuring length and width, respectively.

To keep dA small, we need to keep dw small, as the latter term is getting multiplied by l, which is large. Since dl is getting multiplied by w, which is smaller, dl does not affect dA as much as dw.

So, measure width, i.e., smaller dimension, more accurately.

Prob 50 (in Hw 6)

$$\frac{1}{R} = \frac{1}{\underset{x}{R_1}} + \frac{1}{\underset{y}{R_2}} \quad \text{———} (1) \qquad \begin{matrix} R(R_1, R_2) \\ R(x, y) \end{matrix}$$

$$dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2$$

Do implicit differentiation of (1) w.r.t R_1 and then R_2, then solve for dR

Extreme Values and Saddle Points (Section 13.7)

In 1D: local maxima/minima
and inflection points

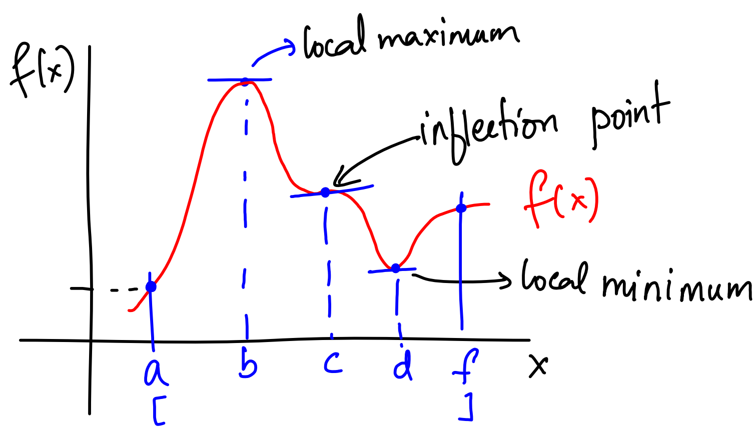
$\rightarrow$ all are **critical points**.

$f'(x) = 0$ at $x = b, c, d$

$f''(x) < 0$ at $x = b \Rightarrow$ local max

$f''(x) = 0$ at $x = c \Rightarrow$ inflection

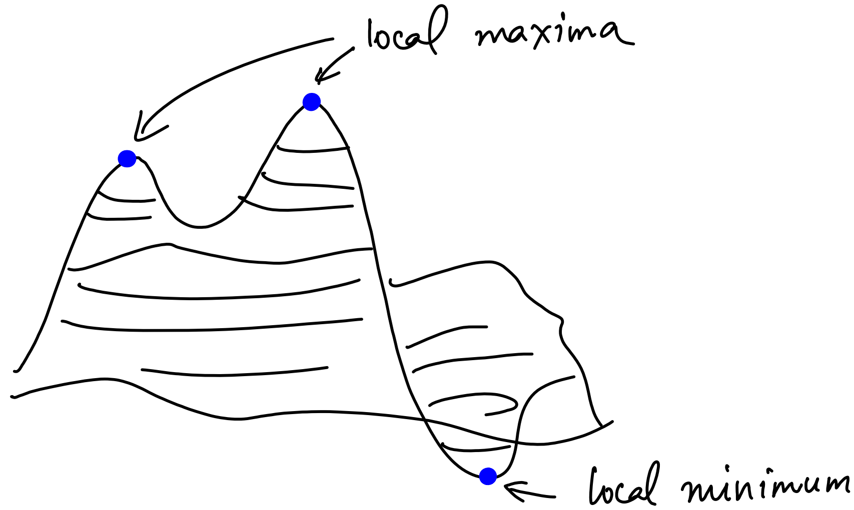
$f''(x) > 0$ at $x = d \Rightarrow$ local min



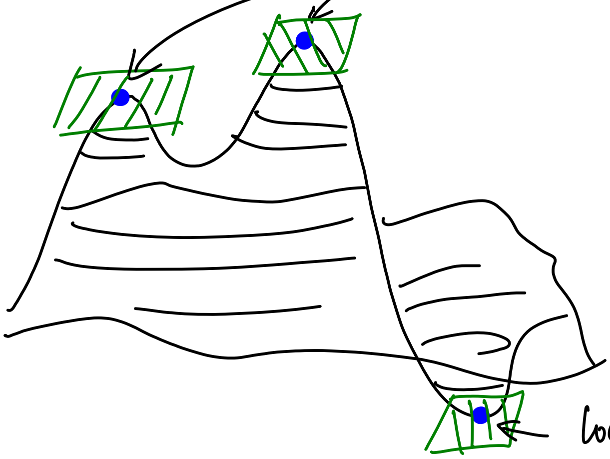
To find all critical points, we also examine the boundary of the domain.

We extend these ideas to 2D and higher dimensions!

Def $f(a,b)$ is a local maximum (local minimum) value of f if $f(a,b) \geq f(x,y)$ $\left(f(a,b) \leq f(x,y)\right)$ for all x,y in an open disk centered at (a,b).



MATH 273 – Lecture 15 (10/14/2014)



Tangent planes at local maxima/minima are horizontal.

Local maxima and local minima are together referred to as local extrema, also called relative extrema.

as opposed to global or absolute extrema.

Indeed, the tangent planes being horizontal at local extrema gives the characterization of these points of interest.

Theorem 10 The first derivative test for local extrema:

If $f(x,y)$ has a local optimum at an interior point (a,b) in its domain, and its first partial derivatives exist at (a,b), then $f_x(a,b)=0$ and $f_y(a,b)=0$.

if f_x or f_y (or both) do not exist, then also (a,b) is a point of interest!

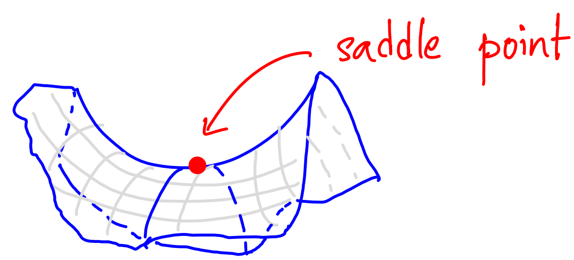
Def An interior point of the domain of $f(x,y)$ where both both f_x and f_y are zero, or where one or both of f_x and f_y do not exist is a **critical point** of $f(x,y)$.

Notice that not all critical points need be local extrema. The partial derivatives could be undefined, or it could be a saddle point, which corresponds to inflection points in 1D.

Def A differentiable function $f(x,y)$ has a **saddle point** at a critical point (a,b) if in every open disk centered at (a,b) there are points (x,y) (in the domain) where $f(x,y) > f(a,b)$, and also other points (u,v) in the domain where $f(u,v) < f(a,b)$.

The corresponding point on the surface $z = f(x,y)$, i.e., $(a,b,f(a,b))$ is a saddle point of the surface.

The name "saddle point" is quite apt.



How do we tell apart the local optima, saddle points, and other critical points?

The Second Derivative Test

Theorem 11 Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b), and $f_x(a,b)=f_y(a,b)=0$. Then

(i) f has a local maximum at (a,b) if
$f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b);

(ii) f has a local minimum at (a,b) if
$f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b);

(iii) f has a saddle point at (a,b) if
$f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b); and

(iv) the test is inconclusive if $f_{xx}f_{yy}-f_{xy}^2=0$ at (a,b).

The quantity $f_{xx}f_{yy}-f_{xy}^2$ is called the **Hessian** or **discriminant** of $f(x,y)$.

Hessian as a 2×2 determinant

$$H(f(x,y)) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Recall, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

Problem 1 Find local extrema and saddle points, if any, of

$$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$$

$f_x = f_y = 0$ for critical points here (notice, domain is all real pairs).

$$f_x = 2x + y + 0 + 3 - 0 + 0 = 0$$

$$f_y = 0 + x + 2y + 0 - 3 + 0 = 0$$

i.e.,

$$2x + y + 3 = 0 \quad \text{——} \quad (1)$$

$$x + 2y - 3 = 0 \quad \text{——} \quad (2)$$

$2\times(1) - (2)$: $3x + 9 = 0$, i.e, $x = -3$, so $y = 3$

So, $(-3,3)$ is the only critical point.

$\frac{\partial}{\partial x}(2x+y+3)$ $\quad$ $\frac{\partial}{\partial y}(x+2y-3)$ $\quad$ $\frac{\partial}{\partial y}(2x+y+3)$

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

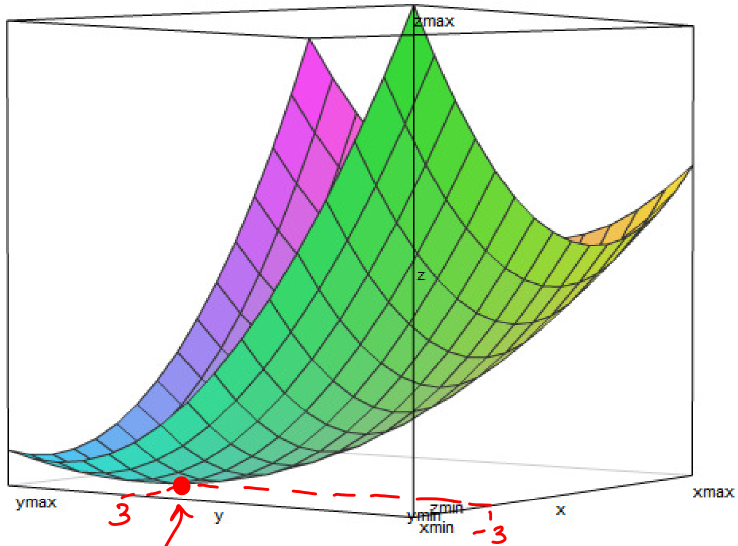
$$H = f_{xx}f_{yy} - f_{xy}^2 = 2\times 2 - (1)^2 = 3 > 0,$$ hence $(-3,3)$ is a local extremum.

Since $f_{xx} = 2 > 0$, $(-3,3)$ is a local minimum of $f(x,y)$.

Also, $f(-3,3) = -5$ is a local minimum of $f(x,y)$.

Let's visualize the surface $z = f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$.

Here, $-6 \le x \le 6$, and
$-6 \le y \le 6$



Indeed, $(-3, 3, -5)$ is a local minimum.

<u>Prob 13</u> $f(x,y) = x^3 - y^3 - 2xy + 6$

Find all local extrema, saddle points, and other critical points of $f(x,y)$.

Domain is all real pairs of values.

$f_x = 3x^2 - 2y = 0$ —— (1)

$f_y = -3y^2 - 2x = 0$ —— (2)

(1) $\Rightarrow$ $y = \frac{3}{2}x^2$. So (2) $\Rightarrow$ $-3\left(\frac{3}{2}x^2\right)^2 - 2x = 0$, i.e.

"implies"

$$x\left(\frac{27}{4}x^3 + 2\right) = 0$$

$x = 0$, or $x^3 = \frac{-8}{27}$, i.e., $x = 0$ or $x = -\frac{2}{3}$, giving

$y = 0$, $y = \frac{3}{2}\left(\frac{-2}{3}\right)^2 = \frac{2}{3}$.

The critical points are $(0,0)$ and $\left(\frac{-2}{3}, \frac{2}{3}\right)$.

$f_{xx} = 6x$, $f_{yy} = -6y$, $f_{xy} = -2$.

$$H = f_{xx}f_{yy} - f_{xy}^2 = (6x)(-6y) - (-2)^2$$
$$= -36xy - 4.$$

$(0,0)$

$H = -36 \cdot 0 \cdot 0 - 4 = -4 < 0$

So $(0,0)$ is a saddle point

$f(0,0) = 6$

$\left(-\frac{2}{3}, \frac{2}{3}\right)$

$H = -36\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) - 4 = 12 > 0$

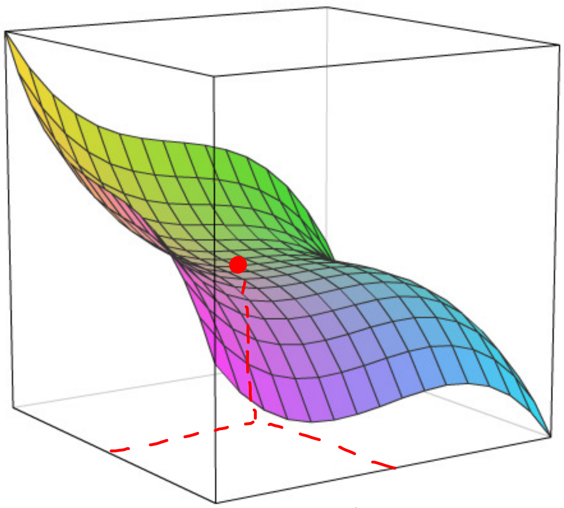
$f_{xx} = 6x = 6\left(\frac{-2}{3}\right) = -4 < 0$

$\left(-\frac{2}{3}, \frac{2}{3}\right)$ is a local maximum.

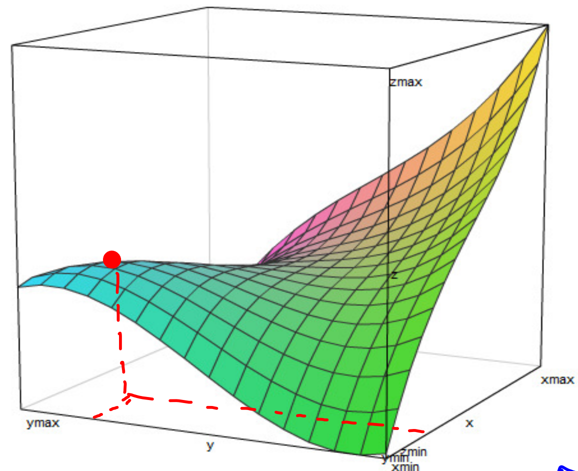
$$f\left(\frac{-2}{3}, \frac{2}{3}\right) = \left(\frac{-2}{3}\right)^3 - \left(\frac{2}{3}\right)^3 - 2\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) + 6$$
$$= \frac{170}{27}.$$

Hence, $(0,0,6)$ is a saddle point on the surface $z = f(x,y) = x^3 - y^3 - 2xy + 6$, while $\left(\frac{-2}{3}, \frac{2}{3}, \frac{170}{27}\right)$ is a local maximum.

Again, let's visualize $z = f(x,y) = x^3 - y^3 - 2xy + 6$.



$(0,0,6)$ – saddle point



$\left(-\frac{2}{3}, \frac{2}{3}, \frac{170}{27}\right)$: local maximum

$-1 \le x \le 1,\ -1 \le y \le 1$

MATH 273 – Lecture 16 (10/16/2014)

Second derivative test

12. $f(x,y) = 1 - \sqrt[3]{x^2+y^2} = 1 - (x^2+y^2)^{1/3}$ → find local extrema, saddle points, and other critical points, if any.

$$f_x = -\frac{1}{3}(x^2+y^2)^{-2/3} \cdot 2x = \frac{-2x}{3(x^2+y^2)^{2/3}} = 0 \quad \text{—— (1)}$$

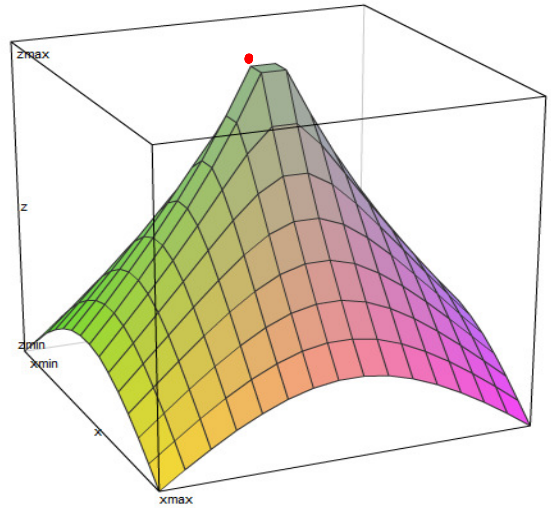
$$f_y = -\frac{1}{3}(x^2+y^2)^{-2/3} \cdot 2y = \frac{-2y}{3(x^2+y^2)^{2/3}} = 0 \quad \text{—— (2)}$$

The only possible solution could have been $(0,0)$, but both f_x and f_y are undefined at $(0,0)$. Hence $(0,0)$ is the only critical point (as f_x, f_y both do not exist at $(0,0)$).

There is no point in finding the second derivatives here (as they also will be undefined at $(0,0)$).

Notice that $f(0,0) = 1$, and $f(x,y) = 1 - \sqrt[3]{x^2+y^2} \leq 1$ for any (x,y). Hence $(0,0)$ is a local maximum.

Here is the surface $z = 1 - \sqrt[3]{x^2+y^2}$. Indeed, the point $(0,0,1)$ is a local maximum. In fact, it is the global maximum!



(15) $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

$$f_x = 12x - 6x^2 + 6y = 0 \quad \text{——} (1)$$

$$f_y = 6y + 6x = 0 \quad \text{——} (2)$$

$(2) \Rightarrow y = -x$. So $(1) \Rightarrow 12x - 6x^2 - 6x = 0$

i.e., $6x - 6x^2 = 0 \qquad 6x(1-x) = 0 \Rightarrow x = 0, 1$

Hence $y = 0, -1$

The critical points are $(0,0)$ and $(1,-1)$.

$f_{xx} = 12 - 12x, \quad f_{yy} = 6, \quad f_{xy} = f_{yx} = 6$

$H = f_{xx} f_{yy} - f_{xy}^2$

(0,0)

$f_{xx} = 12 > 0$

$H = 12 \cdot 6 - 6^2 = 36 > 0$

Hence $(0,0)$ is a local minimum

$f(0,0) = 0.$

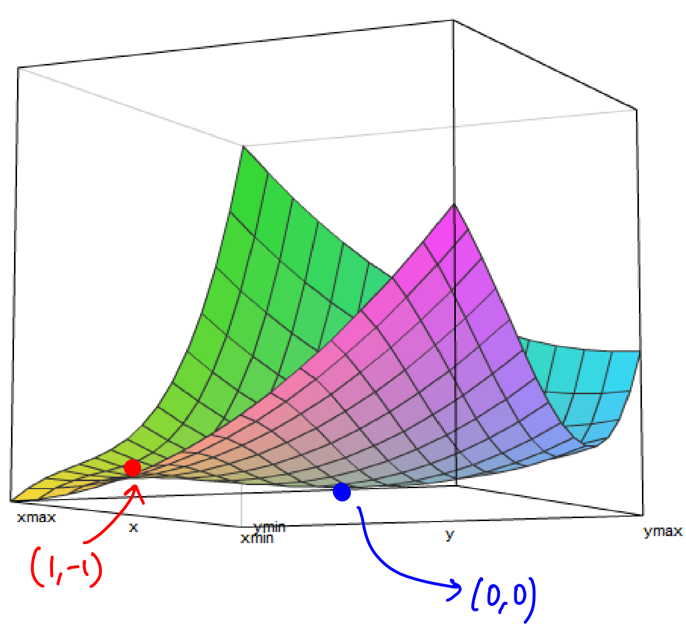
(1,−1)

$f_{xx} = 12 - 12 \times 1 = 0$

$H = \underbrace{0 \times 6 - 6^2}_{f_{xx} \cdot f_{yy} - f_{xy}^2} = -36 < 0$

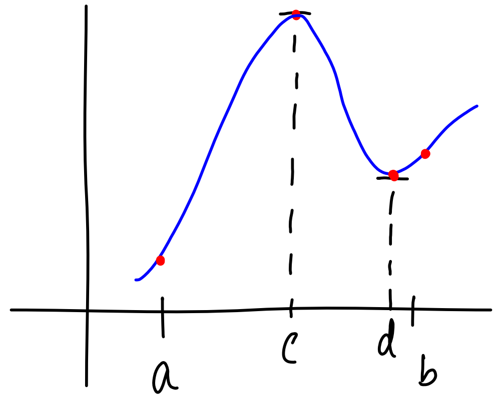
So $(1,-1)$, and hence

$f(1,-1) = 1$ is a saddle point.

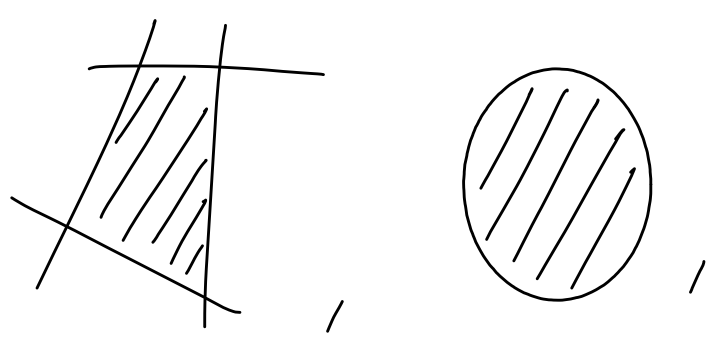


Finding absolute maxima/minima in a region

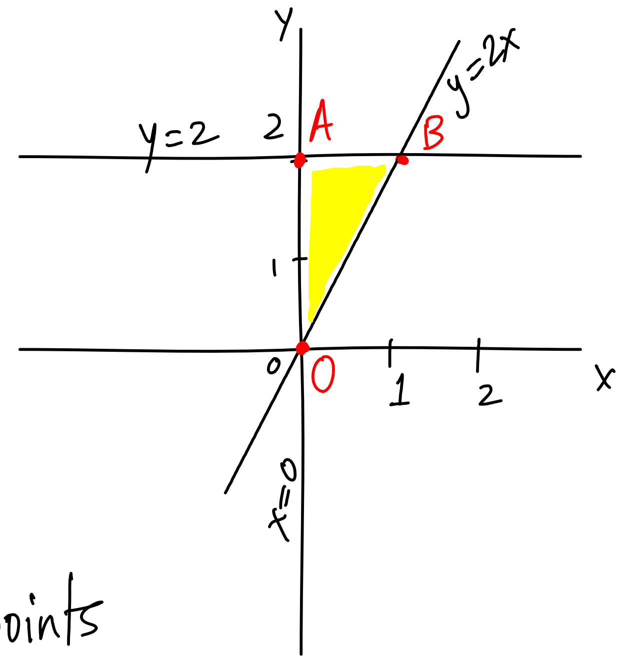
In 1D, to find absolute extrema of $f(x)$ in interval $[a,b]$, we first find local extrema in $[a,b]$, and then compare $f(\cdot)$ at these critical points with $f(a)$ and $f(b)$.



We extend this approach to 2D. In place of interval $[a,b]$, we get a region, typically a closed set in 2D (defined by lines/curves that form its boundaries).

e.g.  , etc.

31. Find the absolute maximum and minimum of $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x=0$, $y=2$, and $y=2x$ in the first quadrant.



The region R is the triangle OAB, where O(0,0), A(0,2), B(1,2).

Step(i) Find any critical points that are interior to R.

$$\left.\begin{aligned} f_x &= 4x-4 = 0 \quad \text{——— (1)} \\ f_y &= 2y-4 = 0 \quad \text{——— (2)} \end{aligned}\right\}$$

$x=1, y=2$. But (1,2) is point B, which is not an interior point.

So, we do not consider the second derivative test at B.

Step(ii) Investigate the behavior of the function on each boundary segment — OA, OB, AB.

OA $x=0$ on OA. Hence

$$f(0,y) = y^2-4y+1$$

Hence, $f'(0,y) = 2y-4=0$, gives $y=2$, which corresponds to $A(0,2)$, an end point.

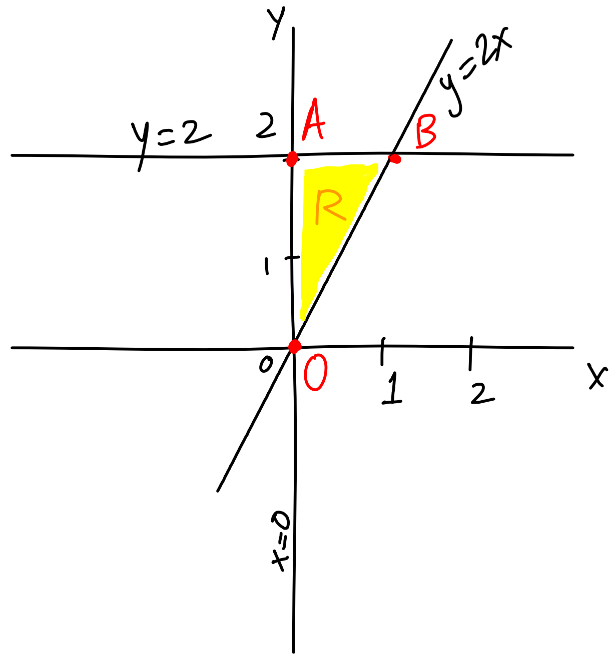
As we did in 1D, we will investigate the function on the boundaries here. But now, we look for critical points on each boundary segment, and also look at the end points. We will finish this problem in the next lecture.

MATH 273 – Lecture 17 (10/21/2014)

Prob 31, continued..

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$

Region R : $\triangle OAB$



(i) No critical points interior to R.
Only critical point (with $f_x = f_y = 0$) is B(1,2), which is on the boundary.

(ii) Check the boundary segments.

on $\underline{OA}$:

$(x=0) \Rightarrow y^2 - 4y + 1 = 0$ giving A(0,2) as the possible critical point, but A is also on the boundary, i.e., at the end of line segment $\overleftrightarrow{OA}$.

on $\underline{AB}$

$y=2$: $f(x,2) = 2x^2 - 4x + (2)^2 - 4(2) + 1 = 2x^2 - 4x - 3$.

$f'(x,2) = 4x - 4 = 0$ giving $x = 1$, and hence B(1,2) is a critical point. But B is not in the middle of $\overleftrightarrow{AB}$.

on $\underline{OB}$

$y=2x$: $f(x,2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1$
$= 6x^2 - 12x + 1$

$f'(x,2x) = 12x - 12 = 0$ giving $x = 1$, so $y = 2\times 1 = 2$.

But B(1,2) is not in the middle of $\overleftrightarrow{OB}$.

(17.2)

We consider $f(x,y)$ at $O(0,0)$, $A(0,2)$, and $B(1,2)$.

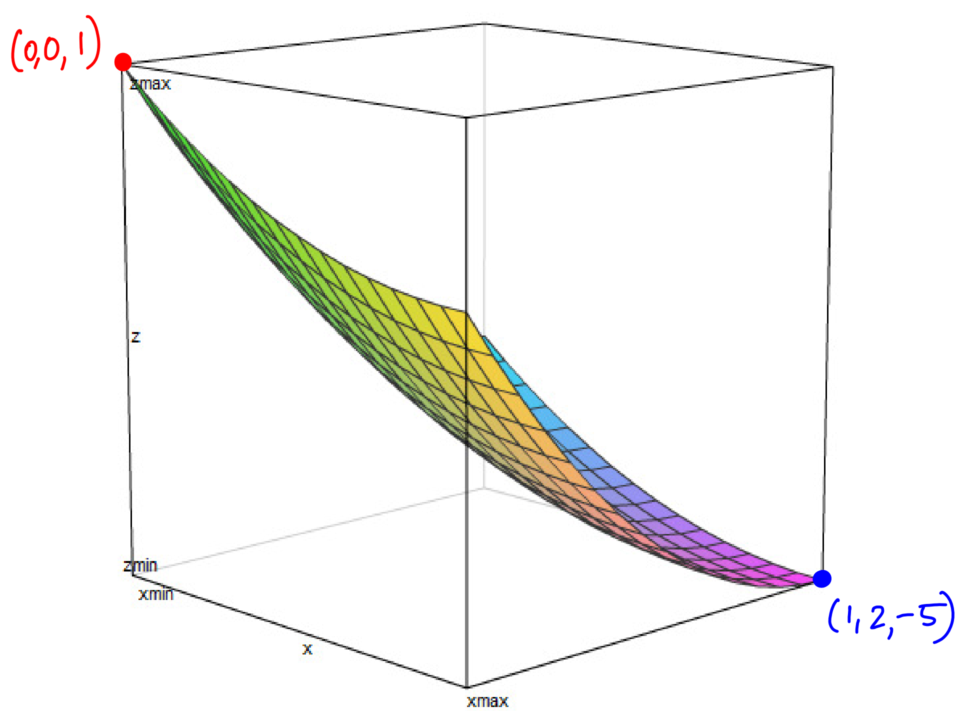
O: $f(0,0) = 1.$ ← absolute maximum in R

A: $f(0,2) = 2(0)^2 - 4(0) + (2)^2 - 4(2) + 1 = -3.$

B: $f(1,2) = 2(1)^2 - 4(1) + (2)^2 - 4(2) + 1 = -5.$ ← absolute minimum in R

The absolute maximum of f in region R occurs at $O(0,0)$, and is $f(0,0) = 1$. The absolute minimum of f in R occurs at $B(1,2)$, and is $f(1,2) = -5$.

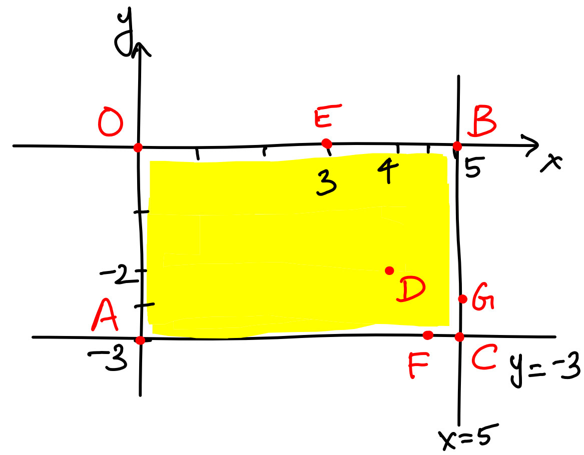
Let's visualize the function in the region R of interest. Below, we set $0 \le x \le 1$, $0 \le y \le 2$.



Notice that the absolute minimum and maximum identified are specific for the region R – they are not the global maximum and minimum over the entire domain.

35. $T(x,y) = x^2 + xy + y^2 - 6x + 2$. Find absolute maximum and absolute minimum of T on the region R that is the rectangular plate defined by $0 \le x \le 5$, $-3 \le y \le 0$.

The corner points are $O(0,0)$, $A(0,-3)$, $B(5,0)$ and $C(5,-3)$.



(i) Look for interior critical points in R.

$$T_x = 2x + y - 6 = 0 \quad \text{——— (1)}$$

$$T_y = x + 2y = 0 \quad \text{——— (2)}$$

$(2) \Rightarrow x = -2y$. So $(1) \Rightarrow$

$$-4y + y - 6 = 0 \Rightarrow -3y = 6$$

i.e., $y = -2$, So $x = -2(-2) = 4$

So $D(4,-2)$ is an interior critical point.

(ii) Examine $T(x,y)$ on boundary segments

<u>OA</u>. $x=0 \Rightarrow T(0,y) = (0)^2 + 0y + y^2 - 6(0) + 2 = y^2 + 2$

$T'(0,y) = 2y = 0$ giving $O(0,0)$, a corner point.

<u>OB</u> $y=0 \Rightarrow T(x,0) = x^2 + x(0) + (0)^2 - 6x + 2 = x^2 - 6x + 2$.

$T'(x,0) = 2x - 6 = 0 \quad x = 3$. So we add $E(3,0)$ to the list of critical points.

AC

$y = 3 \Rightarrow T(x,-3) = x^2 + (-3)x + (-3)^2 - 6x + 2 = x^2 - 9x + 11$

$T'(x,-3) = 2x - 9 = 0$ giving $x = \frac{9}{2}$.

We add $F(9/2, -3)$ to the list of critical points.

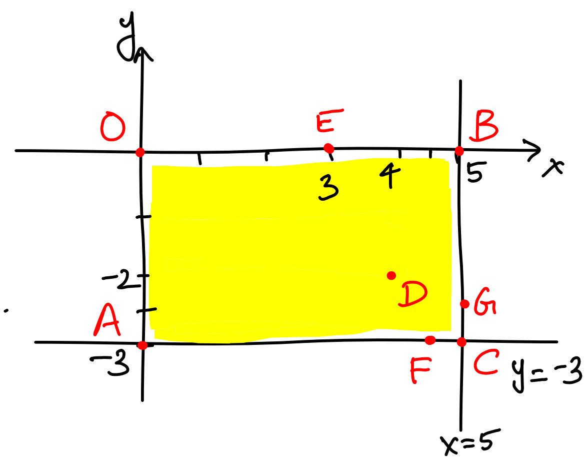
BC

$x=5 \Rightarrow T(5,y) = (5)^2 + (5)y + y^2 - 6(5) + 2 = y^2 + 5y - 3.$

$T'(5,y) = 2y + 5 = 0 \Rightarrow y = -5/2.$

We add $G(5, -5/2)$ to the list of critical points.

We compute f at the eight points (critical or corner points):
$O(0,0)$, $A(0,-3)$, $B(5,0)$, $C(5,-3)$,
$D(4,-2)$, $E(3,0)$, $F(9/2,-3)$, and $G(5,-5/2)$.



$f(x,y) = x^2 + xy + y^2 - 6x + 2$

$O: f(0,0) = 2$	$B: f(5,0) = -3$	$D(4,-2) = -10$	$F(9/2,-3) = -37/4$
$A: f(0,-3) = 11$	$C: f(5,-3) = -9$	$E(3,0) = -7$	$G(5,-5/2) = -37/4$

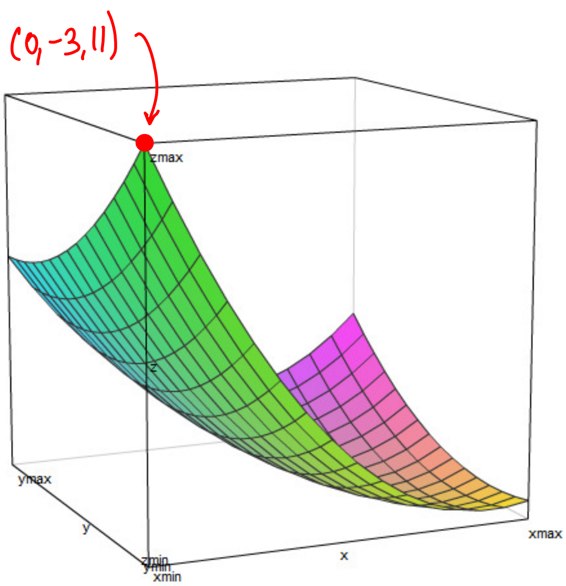
$A(0,-3)$ is the absolute maximum giving $f = 11$, and
$D(4,-2)$ is the absolute minimum, giving $f = -10$.

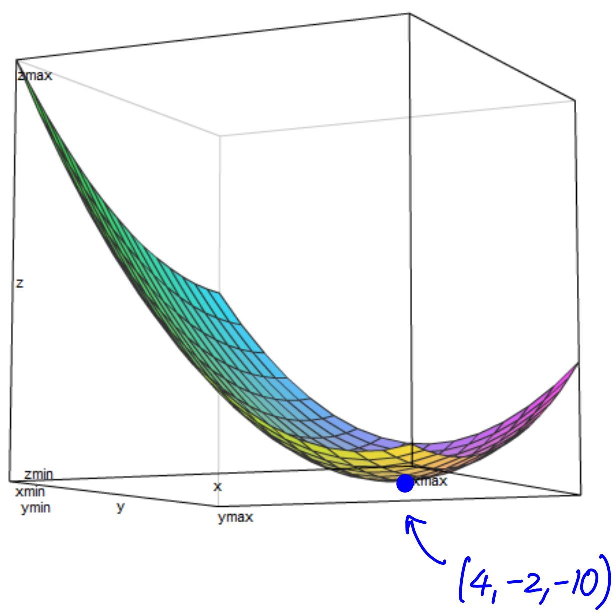
in R.

Notice that we are **not** using the second derivative test to find if a critical point is a local maximum or a local minimum. Irrespective of this information, we would have to compare the value of the function at all these points. So, we do not bother to classify the critical points as local optima, saddle points, etc.

Let's visualize the function $T(x,y)$ over the region R.

$0 \le x \le 5$, $-3 \le y \le 0$ here. Here are two views.





MATH 273 – Lecture 18 (10/23/2014)

Prob 39

Find a, b with $a \le b$ such that

$$\int_a^b (6-x-x^2)\,dx \text{ has its largest value.}$$

let $F(a,b) = \int_a^b (6-x-x^2)\,dx$ with $a \le b$.

$F(a,b)$ is defined for all pairs of real numbers (a,b) satisfying $a \le b$. The line $a=b$ specifies the boundary of the domain.

Indeed, $F(a,a) = \int_a^a (6-x-x^2)\,dx = 0$, So $F(a,b) = 0$ on all the boundary of its domain. So we look for interior critical points.

$$\frac{\partial F}{\partial a} = -(6-a-a^2) = 0 \quad \text{———} (1) \qquad (1) \Rightarrow (a+3)(a-2) = 0$$
$$a = -3, 2$$

$$\frac{\partial F}{\partial b} = (6-b-b^2) = 0 \quad \text{———} (2) \qquad \Rightarrow b = -3, 2$$

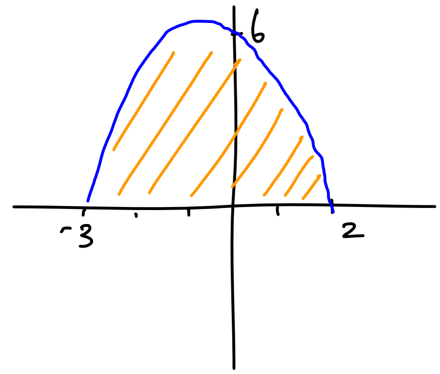
Here, the system of two equations (1) & (2) have independent equations in a and in b, respectively. So, we could consider all possible pairs arising from $a = -3, 2$ and $b = -3, 2$.

But we are looking for interior points satisfying $a \le b$, i.e., $a < b$. So $(-3, 2)$ is the only candidate.

$(-3,-3)$ & $(2,2)$ are boundary points

$$F(-3,2) = \int_{-3}^{2} (6 - x - x^2)\,dx$$

This is the area under the parabola $y = 6 - x - x^2$ between $x = -3$ and $x = 2$ above the x-axis.



This area is strictly bigger than zero, while it is zero on the boundary $(a = b)$.

$$F(3,-2) = 6x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{-3}^{2}$$

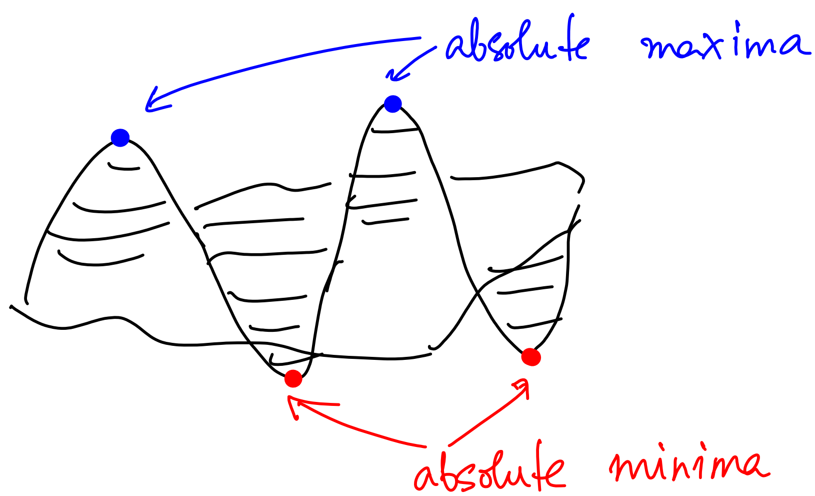
$$= 6(2+3) - \frac{1}{2}(4-9) - \frac{1}{3}(8+27)$$

$$= \frac{180 + 15 - 70}{6} = \frac{125}{6}.$$

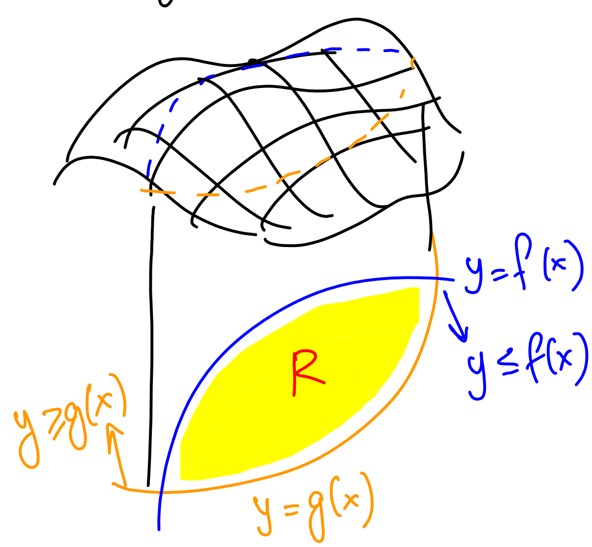
you need not compute $F(a,b)$ for the choice of a, b — the problem asks you to just identify the correct pair a, b.

The other option would be to first compute the integral
$$F(a,b) = \int_a^b (6 - x - x^2)\,dx = 6(b-a) - \frac{1}{2}(b^2 - a^2) - \frac{1}{3}(b^3 - a^3),$$
and then find the absolute maximum of $F(a,b)$ using the techniques we employed previously. The result should be identical.

We could have multiple absolute maxima and/or multiple absolute minima!



The question of finding absolute maxima or absolute minima could be posed over very general regions R (and not just triangles or rectangles).



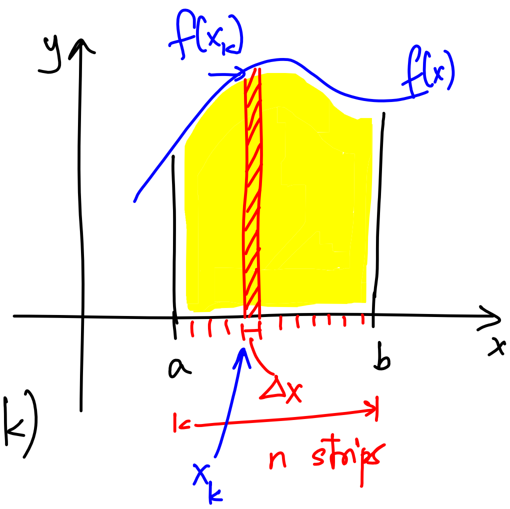
We use the idea of Lagrange multipliers — we'll skip this topic ☺!

Multiple Integrals over Rectangular Domains (Chapter 14)

In 1D

$$\int_a^b f(x)\,dx = \text{Area under curve } y=f(x) \text{ between } x=a \text{ and } x=b \text{ above the } x\text{-axis.}$$

$a \le b$



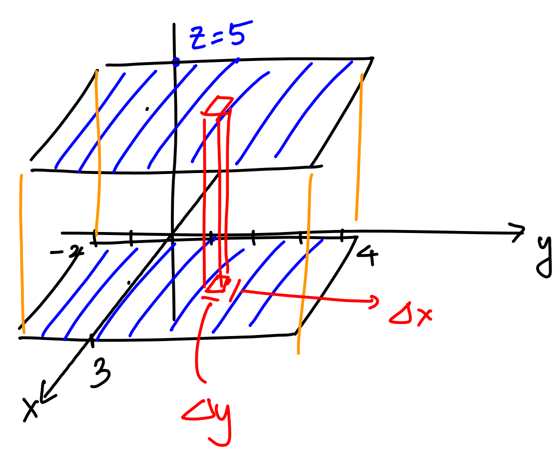
$$= \lim_{n\to\infty} \sum_{k=1}^{n} (\text{area of rectangular strip } k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k)\Delta x$$

We extend this idea to 2D!

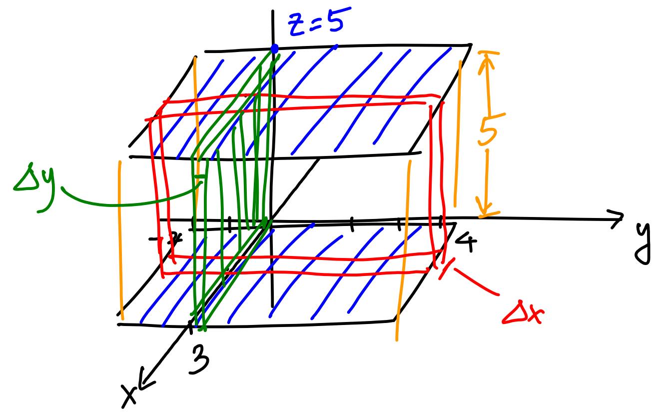
Consider $z=f(x,y)=5$, and the volume under this surface between $0\le x\le 3$, $-2\le y\le 4$, above the xy-plane.



We could evaluate this volume by adding up volumes of thin rectangular columns of widths Δx and Δy.

Volume $V = \lim_{\Delta x \to 0} \sum_{i=1}^{m} 6 \times 5 \times \Delta x \to f(x,y)$

$= \lim_{\Delta y \to 0} \sum_{j=1}^{n} 3 \times 5 \times \Delta y$



We could have $f(x,y)$ be more general — and not necessarily constant, here!

$$V = \lim_{\Delta x, \Delta y \to 0} \sum f(x,y)\, \Delta x \Delta y = \lim_{\Delta A \to 0} \sum f(x,y) \underbrace{\Delta A}_{\text{small area}}$$

The volume under the surface $z = f(x,y)$ within $a \le x \le b$, $c \le y \le d$, above the xy plane is defined as the **double integral**

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

inner

outer

The order of integration is immaterial as long as $f(x,y)$ is continuous over all of R (Fubini's theorem).

MATH 273 – Lecture 19 (10/28/2014)

(Section 14.1)

$$\iint_R f(x,y)\,dA = \int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{\text{inner}}\,dy = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{\text{inner}}\,dx,$$

$R:\ a \le x \le b,\ c \le y \le d$

assuming $f(x,y)$ is continuous over R.

Recall:

$$\frac{\partial^2 f}{\partial x \partial y} = \underbrace{\frac{\partial}{\partial x}}_{\text{outer}}\underbrace{\left(\frac{\partial f}{\partial y}\right)}_{\text{inner}}$$

1. $$\int_1^2\int_2^4 2xy\,dy\,dx = \int_1^2\left[x\int_2^4 2y\,dy\right]dx$$

x-limits

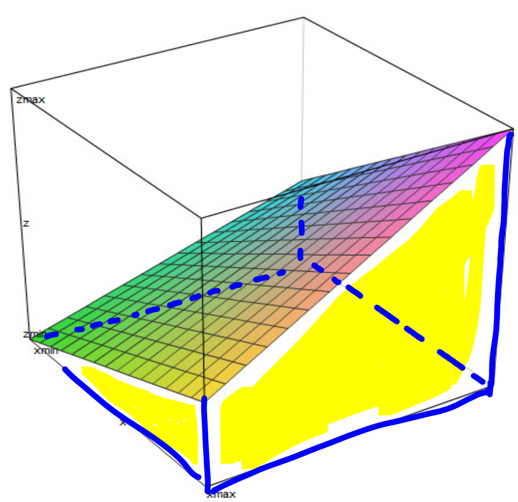
y-limits

book has 0 instead of 2

$$= \int_1^2 x\left(y^2\Big|_2^4\right)dx$$

$$= \int_1^2 x\left(\underbrace{(4)^2}_{16} - \underbrace{(2)^2}_{4}\right)dx = \int_1^2 12x\,dx = 6x^2\Big|_1^2$$

$$= 6\left(\underbrace{(2)^2}_{4} - \underbrace{(1)^2}_{1}\right) = 6 \times 3 = 18.$$



The volume of the solid block under $z = 2xy$ bounded by R.

7. $\int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy$

$$= \int_0^1 \left(\ln|1+xy| \Big|_0^1 \right) dy$$

$$= \int_0^1 \left(\ln|(1+y)| - \underbrace{\cancel{\ln 1}}_{=0} \right) dy = \int_0^1 \ln(|1+y|)\, dy$$

$$= (1+y)\ln|1+y| - (1+y) \Big|_0^1$$

$$= \left[(1+1)\ln(1+1) - (1+1)\right] - \left[1 \overbrace{\cancel{\ln 1}}^{=0} - 1\right]$$

$$= 2\ln 2 - 1.$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \ln x\, dx = x\ln x - x + C$$

$$\frac{d}{dx}(x\ln x - x) = \cancel{x \cdot \frac{1}{x}} + 1 \cdot \ln x - \cancel{1}$$

8. $\int_1^4\int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\,dy = \int_1^4 \left(\left[\frac{1}{4}x^2 + \sqrt{y}\,x\right] \Big|_0^4 \right) dy$

$$= \int_1^4 \left(\frac{1}{4}\left((4)^2 - (0)^2\right) + \sqrt{y}\,(4-0) \right) dy = \int_1^4 (4 + 4\sqrt{y})\, dy$$

$$= 4y + \frac{4y^{3/2}}{(3/2)} \Big|_1^4 = 4(4-1) + \frac{8}{3}\left((4)^{3/2} - (1)^{3/2}\right)$$

$$= 4 \times 3 + \frac{8}{3}(8-1) = 12 + \frac{56}{3} = \frac{92}{3}.$$

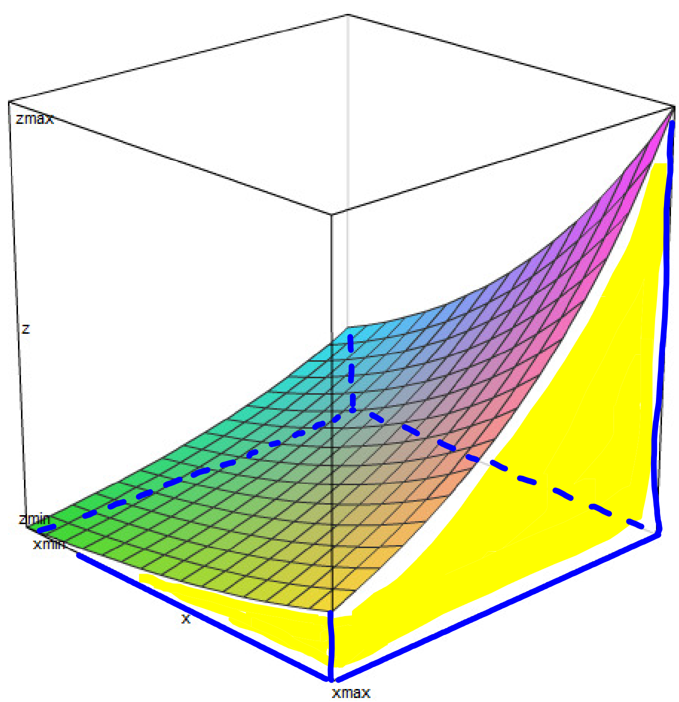
9. $$\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx = \int_0^{\ln 2}\left(\int_1^{\ln 5}(e^{2x})e^{y}\,dy\right)dx$$

$$= \int_0^{\ln 2} e^{2x}\left(e^{y}\Big|_1^{\ln 5}\right)dx = \int_0^{\ln 2} e^{2x}\left(e^{\ln 5}-e^{1}\right)dx$$

$$= \int_0^{\ln 2} e^{2x}(5-e)\,dx = (5-e)\ \frac{1}{2}e^{2x}\Big|_0^{\ln 2}$$

$$= \left(\frac{5-e}{2}\right)\left(\underbrace{e^{2\ln 2}}_{4}-\underbrace{e^{2\times 0}}_{1}\right) = \frac{3}{2}(5-e).$$

$$\frac{d}{dx}\left(e^{2x}\right) = 2.e^{2x}$$



$0 \le x \le \ln 2$, $0 \le y \le \ln 5$ giving $1 \le z \le 20$.

The volume of the solid object bounded above by $z = e^{2x+y}$ and the rectangle $0 \le x \le \ln 2$, $0 \le y \le \ln 5$ is $\frac{3}{2}(5-e)$.

15. Find $\iint\limits_R xy\cos y \, dA$ for $R: -1 \le x \le 1,\ 0 \le y \le \pi$

Do integration w.r.t x first! → The calculation simplifies this way.

$$\int_0^{\pi}\int_{-1}^{1} xy\cos y \, dx\, dy = \int_0^{\pi} y\cos y\left(\frac{1}{2}x^2\Big|_{-1}^{1}\right)dy = \int_0^{\pi}\frac{y\cos y}{2}\big(\underbrace{1^2-(-1)^2}_{=0 \;(1-1)}\big)dy = 0 \ !$$

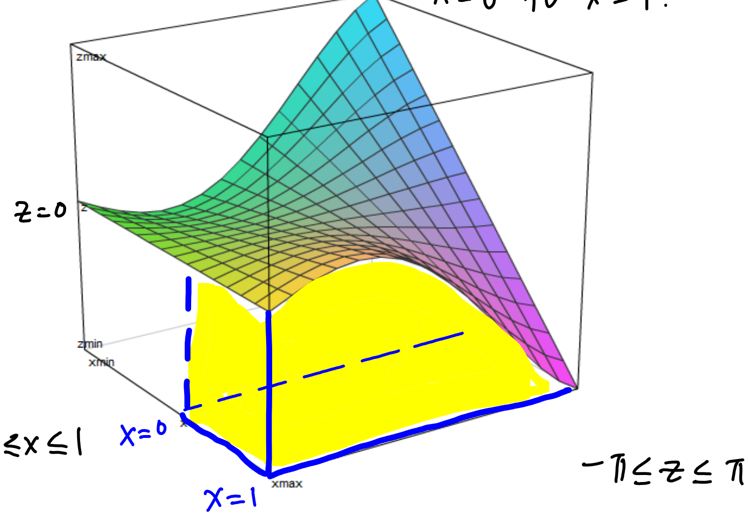
We get the same answer if we integrate w.r.t to y first.

$$\int_{-1}^{1}\int_0^{\pi} xy\cos y\, dy\, dx = \int_{-1}^{1} x\left(\int_0^{\pi} y\cos y\, dy\right)dx = \int_{-1}^{1} x\left[y\sin y + \cos y\Big|_0^{\pi}\right]dx$$

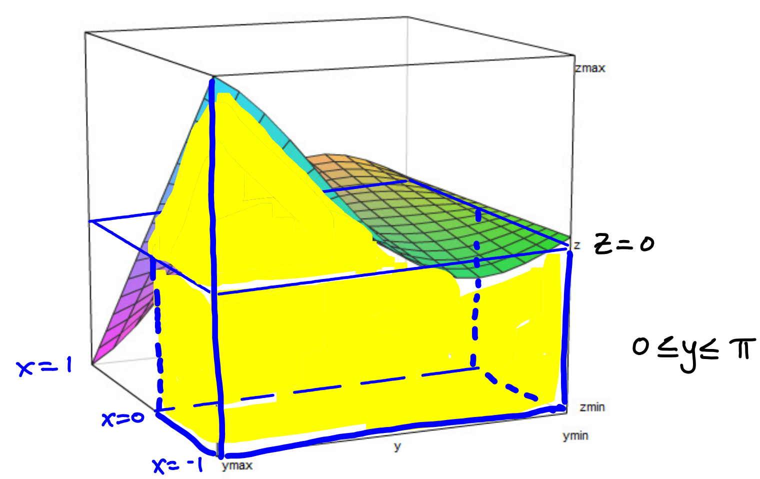
$$= \int_{-1}^{1} x\Big[\pi\underbrace{\sin\pi}_{=0} + \underbrace{\cos\pi}_{-1} - \big(0\underbrace{\sin 0}_{=0} + \underbrace{\cos 0}_{1}\big)\Big]dx = \int_{-1}^{1} -2x\,dx = -x^2\Big|_{-1}^{1} = -(1^2-(-1)^2) = 0.$$

Recall that the double integral computes the volume under the surface $z=f(x,y)$ and above the xy-plane. If the volume is **below the xy-plane**, it is **negative**. In this case, the positive and negative volumes cancel each other.

negative volume (=−1) from $x=0$ to $x=1$.



positive volume (=1) from $x=-1$ to $x=0$.



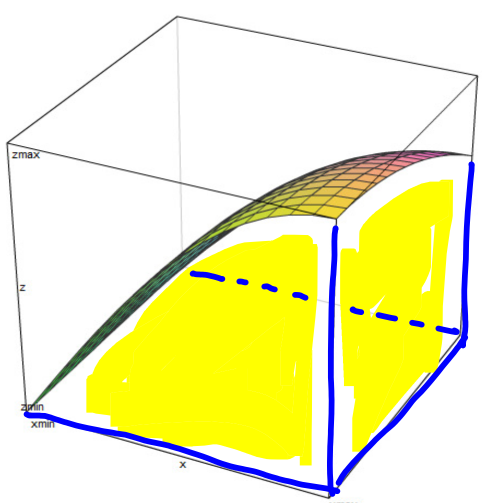
27. Find volume of region bounded above by the surface $z = 2\sin x \cos y$ and below by rectangle R $0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

$$\text{Volume} = \iint_R 2\sin x \cos y \, dA$$

$$= \int_0^{\pi/4}\int_0^{\pi/2} 2\sin x \cos y \, dx\, dy = \int_0^{\pi/4}\left[2\cos y\,(-\cos x)\Big|_0^{\pi/2}\right]dy$$

$$= \int_0^{\pi/4} 2\cos y\Big(\underbrace{-\cos \pi/2}_{=0} - (-\underbrace{\cos 0}_{1})\Big)dy = \int_0^{\pi/4} 2\cos y \, dy$$

$$= 2\sin y\Big|_0^{\pi/4} = 2\Big(\underbrace{\sin \pi/4}_{=\frac{1}{\sqrt{2}}} - \underbrace{\sin 0}_{=0}\Big) = \frac{2}{\sqrt{2}} = \sqrt{2}.$$



$0 \le x \le \pi/2$, $0 \le y \le \frac{\pi}{4}$ here.

MATH 273 – Lecture 20 (10/30/2014)

Double integration over general domains (14.2)

Theorem 2 (Fubini's stronger theorem)

Let $f(x,y)$ be a continuous function on region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with $g_1(x)$ and $g_2(x)$ are continuous over $x \in [a,b]$, then

"element of" or "in"

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with $h_1(y)$ and $h_2(y)$ are continuous over $y \in [c,d]$, then

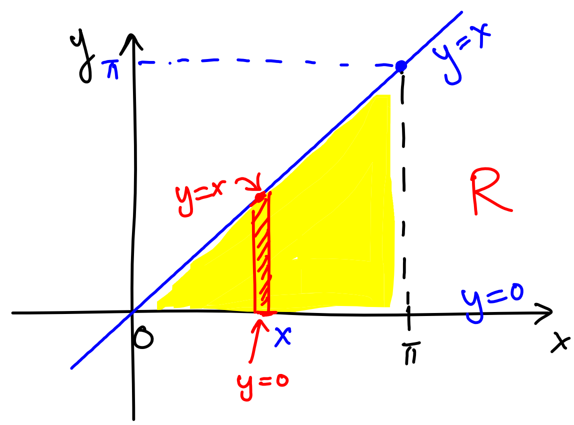
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

For a given integral $\iint_R f(x,y)\,dA$, we could use either of these two forms, and we should get the same answer.

We first try to evaluate such a double integral, and then provide details of how to specify the details of the region of integration.

19. Sketch the region of integration R and evaluate the integral $\int_0^{\pi}\int_0^{x} x\sin y \, dy\, dx$.

The integral is of the form described in Option 1, $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$, with $g_1(x) = 0$ and $g_2(x) = x$. Or, the limits of y are $y=0$ to $y=x$, and limits of x are $[0,\pi]$.



More generally, one needs to plot $g_1(x)$ and $g_2(x)$, and decide which sides of these two curves to pick.

$$\iint_R x\sin y\, dA = \int_0^{\pi}\int_0^{x} x\sin y\, dy\, dx = \int_0^{\pi}\left(x\left(-\cos y\right)\Big|_0^x\right)dx$$

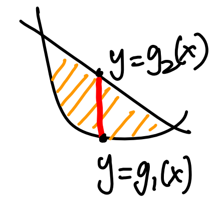
$$= \int_0^{\pi} x\left[-\cos x - -\cos 0\right]dx = \int_0^{\pi} x(1-\cos x)\,dx$$

$$= \int_0^{\pi}(x - x\cos x)\,dx = \left[\frac{1}{2}x^2 - (x\sin x + \cos x)\right]\Big|_0^{\pi}$$

$$= \frac{1}{2}(\pi^2) - \pi\underbrace{\cancel{\sin\pi}}_{=0} - \underbrace{\cos\pi}_{-1} - \left(\frac{1}{2}(0^2) + 0\sin 0 + \underbrace{\cos 0}_{1}\right) = \frac{\pi^2}{2} + 2.$$

Sketching Regions of Integration

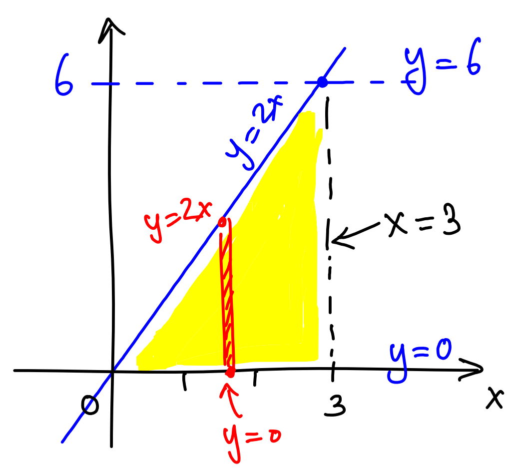
Procedure using vertical cross sections



1. Sketch region and label bounding curves.
2. Imagine a vertical line crossing the region at x, and figure out the limits of y as functions of x.
3. Find the limits for x, such that the region includes all possible vertical lines as used in Step 2.

The procedure using horizontal cross sections is similar, except that the roles of x and y are reversed.

(1). Sketch the region of integration $0 \le x \le 3$, $0 \le y \le 2x$.

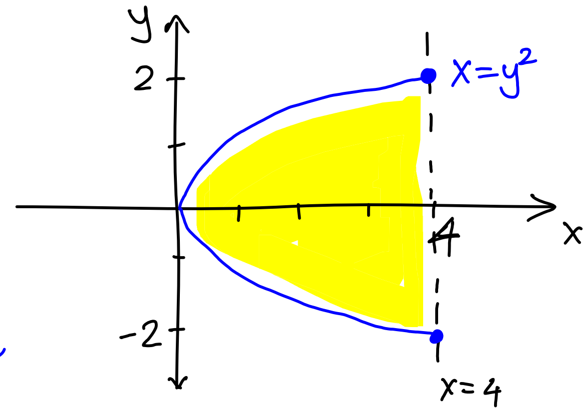


Since the limits of y are given as functions of x here, we are indeed using vertical cross sections. But notice that we could equivalently describe the region as

$$0 \le y \le 6, \ \frac{y}{2} \le x \le 3;$$

using horizontal cross sections.

3. $-2 \le y \le 2$, $y^2 \le x \le 4$

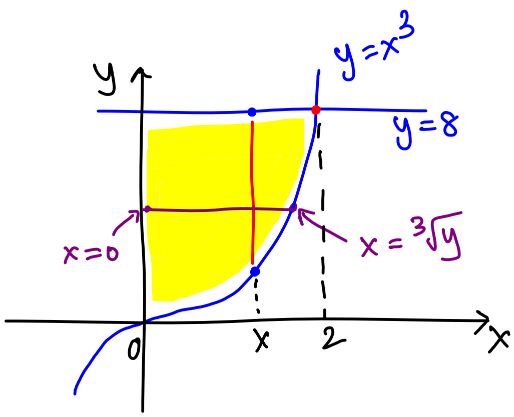


$x=y^2$ gives

$y=\sqrt{x}$

$x=y^2$ has the shape of the parabola $y=x^2$, but with x and y flipped.

(9). Write the integral for $\iint_R dA$ over region R using (a) vertical cross sections and (b) horizontal cross sections.



(a) $\displaystyle\int_0^2\int_{x^3}^8 dy\,dx$

Notice that for the vertical line cutting across the region, y varies from x^3 to 8.

Also, $y=x^3$ and $y=8$ intersect at $(2,8)$.

(b) $\displaystyle\int_0^8\int_0^{\sqrt[3]{y}} dx\,dy$

For the horizontal line crossing the region, x varies from 0 to $(y)^{1/3}$, i.e., $\sqrt[3]{y}$.

MATH 273 – Lecture 21 (11/04/2014)

Exam 2: Next Thursday (Nov 13)
– Practice Exam 2 will be posted.

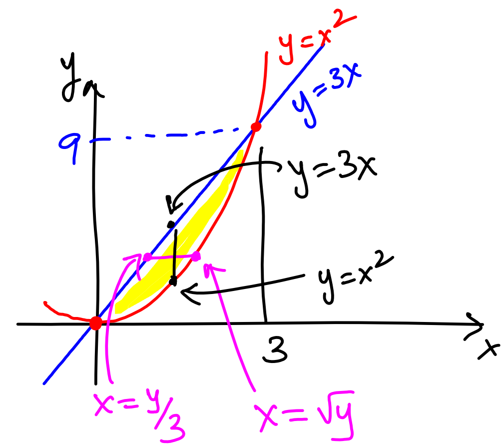
11. Write an integral for $\iint_R dA$ over region R using (a) vertical cross sections, and (b) horizontal cross section.

$y = 3x$, $y = x^2$ — points of intersection of these two curves

$x^2 = 3x$

$x^2 - 3x = 0 \Rightarrow x(x-3) = 0$

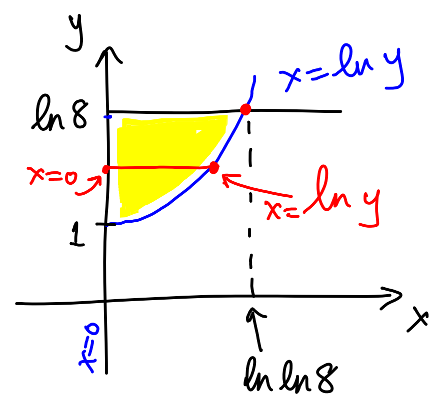
gives $x = 0$, $x = 3$, for which $y = 0$, $y = 9$, i.e., the points are $(0,0)$ and $(3,9)$.



(a) $\displaystyle\int_0^3 \int_{x^2}^{3x} dy\, dx$ (b) $\displaystyle\int_0^9 \int_{y/3}^{\sqrt{y}} dx\, dy$

21. Sketch the region of integration, and evaluate the integral.

$$\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx\, dy$$



plot $x = \ln y$, i.e., $y = e^x$

The right point of intersection has $x = \ln(\ln 8) = \ln \ln 8$.

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy = \int_1^{\ln 8} e^y\left(\int_0^{\ln y} e^x dx\right)dy = \int_1^{\ln 8}\left(e^{y+x}\Big|_0^{\ln y}\right)dy$$

$$= \int_1^{\ln 8}\left(e^{y+\ln y} - e^{y+0}\right)dy = \int_1^{\ln 8}\left(e^y\cdot \underbrace{e^{\ln y}}_{y} - e^y\right)dy$$

$$= \int_1^{\ln 8}\left(ye^y - e^y\right)dy = \left[\underbrace{ye^y - e^y}_{\frac{d}{dy}(ye^y-e^y)=ye^y} - e^y\right]\Bigg|_1^{\ln 8}$$

$$= \left(\ln 8\, e^{\ln 8} - 2e^{\ln 8}\right) - \left(e^1 - 2e^1\right)$$

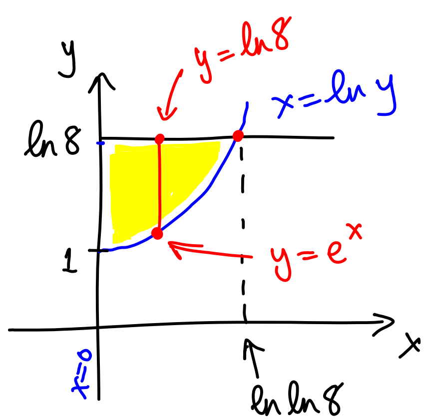
$$= 8\ln 8 - 16 + e = e + 8(\ln 8 - 2).$$

$$\frac{d(uv)}{dy} = u\frac{dv}{dy} + v\frac{du}{dy}$$

or

$$d(uv) = u\,dv + v\,du$$

Let's evaluate the integral now by reversing the order of integration.



$$\int_0^{\ln\ln 8}\int_{e^x}^{\ln 8} e^{x+y}\,dy\,dx = \int_0^{\ln\ln 8}\left(e^{x+y}\Big|_{e^x}^{\ln 8}\right)dx$$

$$= \int_0^{\ln\ln 8}\left(e^{x+\ln 8} - e^{x+e^x}\right)dx$$

$$= \int_0^{\ln\ln 8}\left(8e^x - e^x\cdot e^{e^x}\right)dx$$

$$= \int_0^{\ln\ln 8} \left(8e^x - e^x \cdot e^{e^x}\right) dx = 8e^x - e^{e^x} \Big|_0^{\ln\ln 8}$$

recall: $\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$

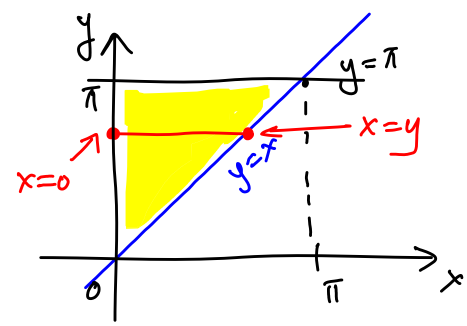
$$= \left(8e^{\ln\ln 8} - e^{e^{\ln\ln 8}}\right) - \left(8e^0 - e^{e^0}\right)$$

$e^{\ln 8} = 8$

$$= 8\ln 8 - 8 - 8 + e = e + 8(\ln 8 - 2).$$

47. Sketch region of integration, reverse the order of integration, and evaluate the integral.

$$I = \int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$



Originally, vertical cross sections are used. We reverse to use horizontal cross sections.

$$I = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy$$

$$= \int_0^{\pi}\left(\frac{\sin y}{y}\, x\Big|_0^y\right) dy = \int_0^{\pi}\left(\frac{\sin y}{y}(y-0)\right) dy = \int_0^{\pi} \sin y\, dy$$

$$= -\cos y\Big|_0^{\pi} = 1 - -1 = 2.$$

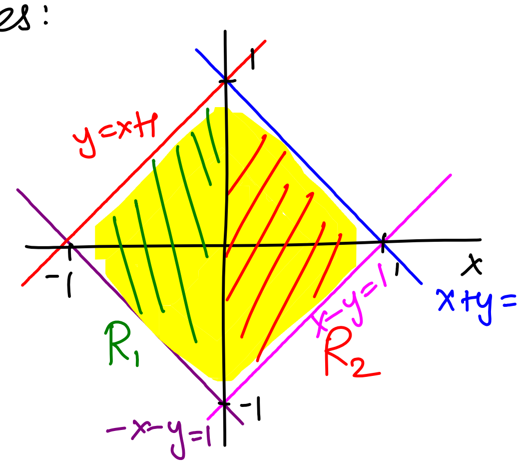
In all the integrals we have seen so far, the region of integration R is bounded essentially by two curves. Notice that even in the case where R is a triangle, as seen in the example above, $y=x$ and $y=\pi$ were sufficient to describe it, along with $x=0$. Now we consider more general regions R, which we split into component regions R_1, R_2, R_3, etc., where each component region is simpler, just as we have seen so far.

55. Find $I = \iint_R (y-2x^2)\,dA$ where R is the region bounded by the square $|x|+|y|=1$.

$|x|+|y|=1$ splits into four lines:

$\pm x \pm y = 1$, i.e.,

$$\begin{aligned} x+y&=1 \\ x-y&=1 \\ -x+y&=1 \\ -x-y&=1 \end{aligned}$$



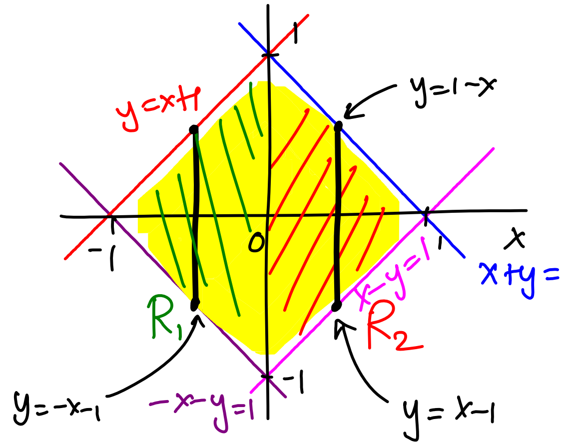
The region is bounded by 4 curves, instead of 2. So split into two regions bounded by two curves each.

... we'll finish this problem in the next lecture ...

MATH 273 – Lecture 22 (11/06/2014)

55. $I = \iint_R (y-2x^2)\,dA$

$$= \iint_{R_1} (y-2x^2)\,dA + \iint_{R_2} (y-2x^2)\,dA$$



Can use vertical cross sections to write each integral.

$$I = \int_{-1}^{0}\int_{-x-1}^{x+1} (y-2x^2)\,dy\,dx + \int_{0}^{1}\int_{x-1}^{1-x} (y-2x^2)\,dy\,dx$$

$$= \int_{-1}^{0} \left(\tfrac{1}{2}y^2 - 2x^2 y \Big|_{\underbrace{-x-1}_{=-(x+1)}}^{x+1}\right) dx + \int_{0}^{1} \left(\tfrac{1}{2}y^2 - 2x^2 y \Big|_{\underbrace{x-1}_{=-(1-x)}}^{1-x}\right) dx$$

$$= \int_{-1}^{0} \left(\tfrac{1}{2}\left[\cancel{(x+1)^2} - \cancel{(-(x+1))^2}\right] - 2x^2 \left[\underbrace{(x+1) - -(x+1)}_{2(x+1)}\right]\right) dx$$

$$+ \int_{0}^{1} \left(\tfrac{1}{2}\left[\cancel{(1-x)^2} - \cancel{(-(1-x))^2}\right] - 2x^2 \left[\underbrace{1-x - -(1-x)}_{2(1-x)}\right]\right) dx$$

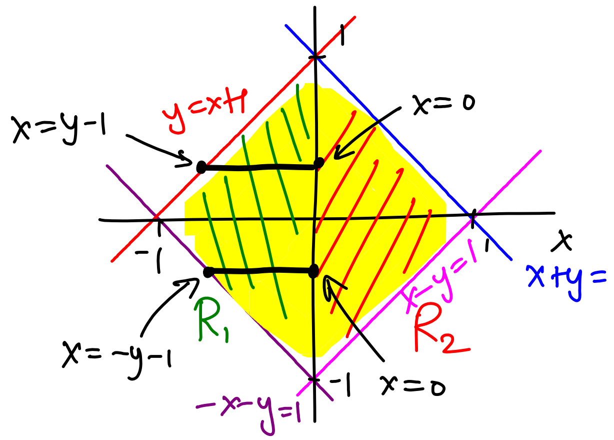
$$= \int_{-1}^{0} -4(x^3+x^2)\,dx + \int_{0}^{1} -4(x^2-x^3)\,dx$$

$$= \left(-x^4 - \tfrac{4}{3}x^3\right)\Big|_{-1}^{0} + \left(-\tfrac{4}{3}x^3 + x^4\right)\Big|_{0}^{1} = -\left(-(-1)^4 - \tfrac{4}{3}(-1)^3\right)$$

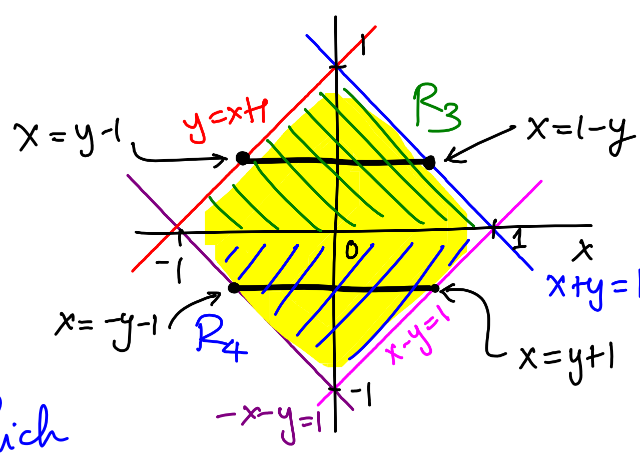
$$+ -\tfrac{4}{3}(1)^3 + (1)^4$$

$$= 1 - \frac{4}{3} - \frac{4}{3} + 1 = \frac{-2}{3}.$$

We might not want to use horizontal cross sections after splitting R into R_1 and R_2 as we did.



But instead, we could have split R horizontally into R_3 and R_4, say, and then used horizontal cross sections.



Just as we could choose which variable to differentiate first w.r.t. in a second derivative, e.g., $\frac{\partial^2 f}{\partial x \partial y}$, we could choose which variable to integrate w.r.t. in a double integral, so that the computation becomes easier.

Properties of Double Integrals

Let $f(x,y)$ and $g(x,y)$ be continuous functions over region R.

1. $$\iint_R c\, f(x,y)\,dA = c\iint_R f(x,y)\,dA$$ (constant multiple)

2. Sum/difference

$$\iint_R \big(f(x,y) \pm g(x,y)\big)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

3. Domination.

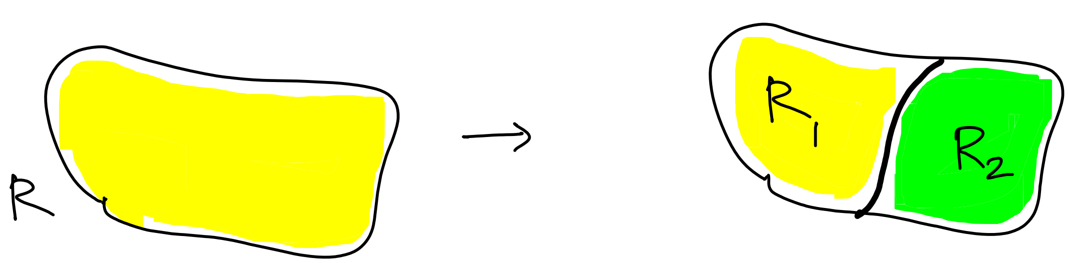
If $f(x,y) \geq g(x,y)$ on R on all points in R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

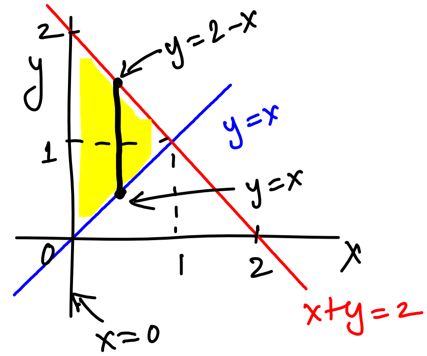
4. Additivity

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$ where

R is the union of nonoverlapping regions R_1 and R_2.



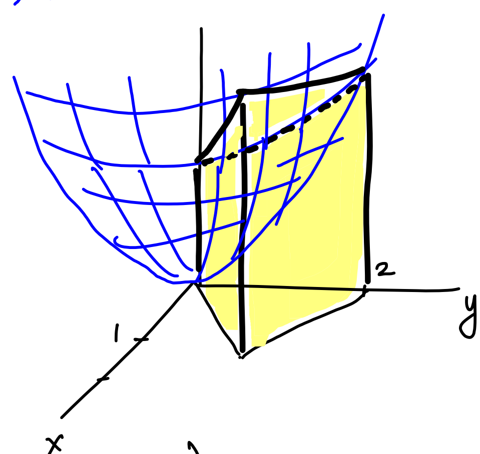
57. Find the volume of the region bounded above by the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the x-y plane.



$$V=\int_0^1\int_x^{2-x}(x^2+y^2)\,dy\,dx = \int_0^1\left(x^2y+\frac{1}{3}y^3\Big|_x^{2-x}\right)dx$$

$$=\int_0^1\left(x^2(\underbrace{2-x-x}_{2-2x})+\frac{1}{3}\left[(2-x)^3-x^3\right]\right)dx$$

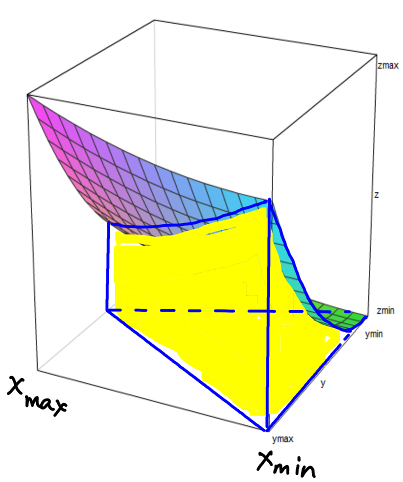
$(2-x)^3 = 8-x^3-12x+6x^2 \longrightarrow (a-b)^3=a^3-3a^2b+3ab^2-b^3$



$$=\int_0^1\left(\frac{3}{3}(2x^2-2x^3)+\frac{1}{3}\left(8-12x+6x^2-2x^3\right)\right)dx$$

$$=\frac{1}{3}\int_0^1\left(8-12x+12x^2-8x^3\right)dx$$

$$=\frac{1}{3}\left[8x-6x^2+4x^3-2x^4\right]\Big|_0^1 = \frac{1}{3}(8-6+4-2)=\frac{4}{3}.$$



$0\le x\le 1$, $0\le y\le 2$ here, giving $0\le z\le 5$.

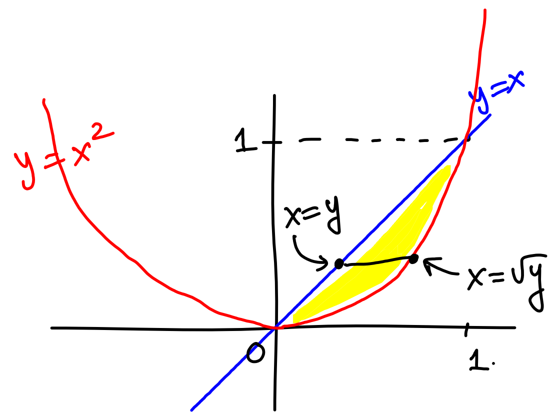
This is a more accurate figure than the one drawn by hand above ☺!

MATH 273 – Lecture 23 (11/11/2014)

Exam 2 Review

7. **(14)** Sketch the region of integration, and write an equivalent integral with the order of integration reversed. Then evaluate this reverse ordered integral.

$$I = \int_0^1 \int_{x^2}^{x} \sqrt{x}\, dy dx.$$



I uses vertical cross sections.
y varies from x^2 to x, and x varies from 0 to 1.

Points of intersection of $y=x$ and $y=x^2$: $x=x^2$, i.e. $x(x-1)=0$, giving $x=0,1$, for which $y=0,1$.
The points of intersection are $(0,0)$ and $(1,1)$.

Reversing the order of integration, we write

$$I = \int_0^1 \int_y^{\sqrt{y}} \sqrt{x}\, dx\, dy.$$

$$= \int_0^1 \left(\frac{x^{3/2}}{(3/2)} \Bigg|_y^{\sqrt{y}} \right) dy = \frac{2}{3} \int_0^1 \left((\sqrt{y})^{3/2} - (y)^{3/2} \right) dy$$

$$= \frac{2}{3} \left[\frac{4}{7} y^{7/4} - \frac{2}{5} y^{5/2} \Bigg|_0^1 \right] = \frac{2}{3} \left[\frac{4}{7}(1) - \frac{2}{5}(1) \right] - 0$$

$$= \frac{2}{3} \left(\frac{4\times 5 - 2\times 7}{7 \times 5} \right) = \frac{2 \times \cancel{6}\,2}{\cancel{3} \times 35} = \frac{4}{35}.$$

23-2

6. **(12)** Evaluate the double integral over the given region R.

$$I = \iint_R xye^{xy^2}\,dA, \qquad R:\ 0 \le x \le 2,\quad 0 \le y \le 1.$$

$$I = \int_0^2\int_0^1 xy\,e^{xy^2}\,dy\,dx$$

$$= \int_0^2 \left(\frac{1}{2}e^{xy^2}\Big|_0^1\right)dx$$

$$= \frac{1}{2}\int_0^2 \left[e^{x(1)^2} - e^{x(0)}\right]dx = \frac{1}{2}\int_0^2 (e^x - 1)\,dx$$

$$= \frac{1}{2}(e^x - x)\Big|_0^2 = \frac{1}{2}\left[e^2 - 2 - (\cancel{e^0}^{\,1} - 0)\right] = \frac{1}{2}(e^2 - 3).$$

Notice that

$$\frac{\partial}{\partial y}\left(e^{xy^2}\right) = e^{xy^2}\frac{\partial}{\partial y}(xy^2)$$
$$= x(2y)\cdot e^{xy^2}$$
$$= 2xy\,e^{xy^2}$$

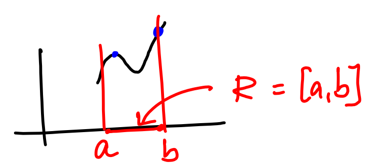
Notice $\frac{\partial}{\partial x}\left(e^{xy^2}\right) = e^{xy^2}\cdot y^2 = y^2e^{xy^2} \longrightarrow$ so, integrating first w.r.t. x is much harder here!

8. **(6)** Decide whether each of the following statements is *True* or *False*. **Justify** your answer.

(a) A point that gives the absolute maximum of a function in a given region R must also be a local maximum of the function.

(b) Swapping the lower and upper limits of both integrals in a double integral leaves the value of the double integral unchanged.

(a) FALSE. The absolute maximum could occur on the boundary of R.



(b) TRUE. Each swap multiplies the integral by -1, so the value is unchanged as $(-1)\times(-1) = 1$.

23-3

3. **(12)** Let $y = uv$. If u is measured with an error of 2% and v is measured with an error of 3%, estimate the percentage error in the calculated value of y.

$y = uv$

The total differential of y is

$\frac{1}{y}\left(dy = u\,dv + v\,du\right).$

We want $\frac{dy}{y}$, given $\frac{du}{u} = 2\%$, $\frac{dv}{v} = 3\%$

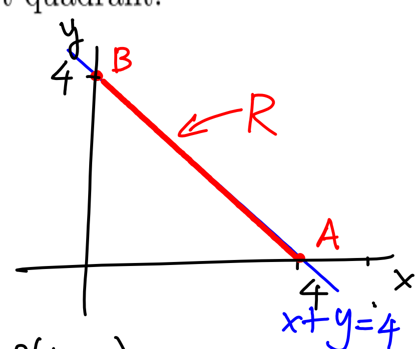
$$\frac{dy}{y} = \frac{u\,dv}{y} + \frac{v\,du}{y}$$

Equivalently, $\frac{du}{u} \times 100 = 2$, $\frac{dv}{v} \times 100 = 3.$

$y = uv$ gives

$$\frac{dy}{y} = \frac{\cancel{u}\,dv}{\cancel{u}v} + \frac{\cancel{v}\,du}{u\cancel{v}} = \frac{dv}{v} + \frac{du}{u} = 3\% + 2\% = 5\%.$$

5. **(16)** Find the absolute maximum and minimum values of $f(x, y) = x^2 + xy + y^2 - 3x + 3y$ on the region R that is the part of the line $x + y = 4$ lying in the first quadrant.



R cannot have interior critical points.

R is $\overline{AB}$ from $A(4,0)$ to $B(0,4)$.

On $\overline{AB}$, $y = 4 - x$, hence

$$f(x, 4-x) = f(x) = x^2 + x(4-x) + (4-x)^2 - 3x + 3(4-x)$$
$$= \cancel{x^2} + 4x - \cancel{x^2} + x^2 - 8x + 16 - 3x + 12 - 3x$$
$$= x^2 - 10x + 28$$

$f'(x) = 2x - 10 = 0$ gives $x = 5$, giving $y = 4 - 5 = -1$.

But $(5,-1)$ is not on $\overline{AB}$. So we just check $f(x,y)$ at $A(4,0)$ and $B(0,4)$.

23-4

$$f(x,y) = x^2 + xy + y^2 - 3x + 3y$$

A: $f(4,0) = (4)^2 + 0 + 0 - 3(4) + 0 = 4$ ← absolute minimum

B: $f(0,4) = (0)^2 + 0 + (4)^2 - 0 + 3\times(4) = 28$ ← absolute maximum.

4. **(14)** Find all local minima, local maxima, and saddle points of the function given below. You should evaluate the function at each critical point.

$$f(x,y) = x^3 + y^3 - 3xy + 15.$$

The domain is all of $\mathbb{R}^2$ (all real pairs).

Critical points $f_x = 3x^2 - 3y = 0$ ——— (1)

$f_y = 3y^2 - 3x = 0$ ——— (2)

(2) $\Rightarrow$ $x = y^2$. Plugging into (1) gives

$3(y^2)^2 - 3y = 0 \Rightarrow y^4 - y = 0 \quad y(y^3 - 1) = 0,$

giving $y = 0, 1$, and hence $x = 0, 1$, So the critical points are $(0,0)$ and $(1,1)$.

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$; So

$H = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9.$

$(0,0)$

$H = 36(0)(0) - 9 = -9 < 0$

$\Rightarrow (0,0)$ is a saddle point.

$f(0,0) = 15.$

$(0,0,15)$ is a saddle point.

$(1,1)$

$H = 36(1)(1) - 9 = 27 > 0.$

$f_{xx} = 6(1) = 6 > 0. \Rightarrow (1,1)$ is a local minimum.

$f(1,1) = (1)^3 + (1)^3 - 3(1)(1) + 15 = 14.$

$(1,1,14)$ is a local minimum.

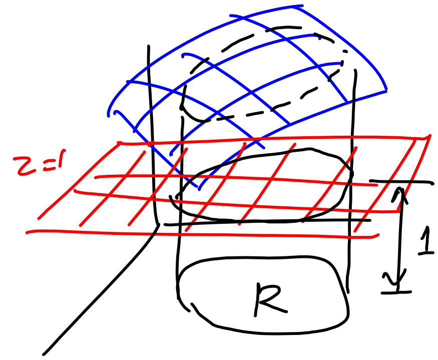
MATH 273 – Lecture 24 (11/18/2014)

Same offer for final as I made for Exam 2
– If you do really well in the Final, its score can replace (to a large extent) the lower scores of Exams 1 and 2.

Area by Double Integration

We saw $\iint_R f(x,y)\,dA$ gives the volume bounded by $z=f(x,y)$ surface above and R on the xy plane below.

What if $z=f(x,y)=1$?



Volume $V=\iint_R 1\,dA = \text{Area}\times 1$

Volume of the 3D solid when $f(x,y)=1$ is just the area $\times$ height

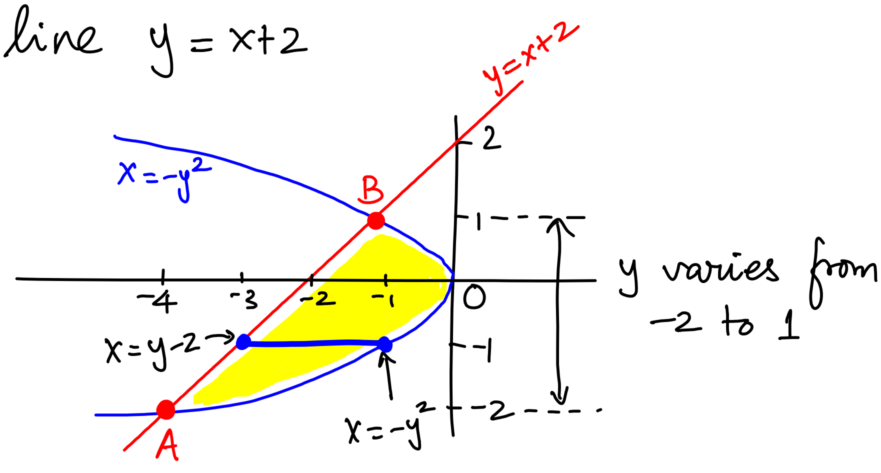
Hence $\text{Area} = \iint_R dA$.

The area of a closed bounded region R in the plane is

$$A=\iint_R dA.$$

③ Sketch the region R bounded by given lines and curves, express its area as a double integral, and evaluate it to find the area.

Parabola $x=-y^2$, line $y=x+2$



Points of intersection:

Plug $x=-y^2$ into $y=x+2$

$$y=-y^2+2$$

$$y^2+y-2=0$$

$$(y+2)(y-1)=0 \Rightarrow y=-2,\ y=1$$

$$\Rightarrow x=-4,\ -1$$

$A(-4,-2)$ and $B(-1,1)$

$$A=\int_{-2}^{1}\int_{y-2}^{-y^2} dx\,dy = \int_{-2}^{1}\left(x\Big|_{y-2}^{-y^2}\right)dy = \int_{-2}^{1}\left[-y^2-(y-2)\right]dy$$

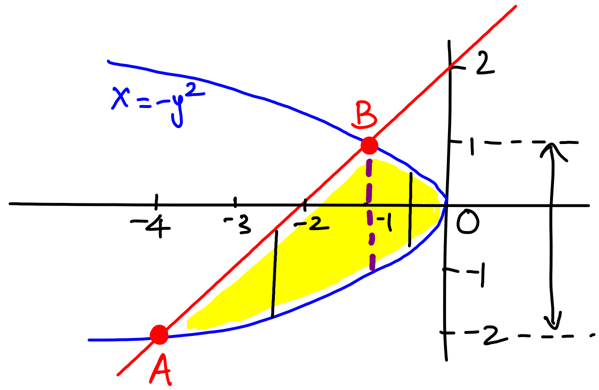
$$=\int_{-2}^{1}(2-y-y^2)\,dy = 2y-\frac{1}{2}y^2-\frac{1}{3}y^3\Big|_{-2}^{1}$$

$$=2\underbrace{(1--2)}_{3}-\frac{1}{2}\underbrace{\left(1^2-(-2)^2\right)}_{-3}-\frac{1}{3}\underbrace{\left(1^3-(-2)^3\right)}_{9}$$

$$=2\times 3+\frac{3}{2}-\frac{9}{3}=$$

$$=6+\frac{3}{2}-3=\frac{9}{2}.$$

Notice that using vertical cross sections to evaluate the integral would require a split of the region into two



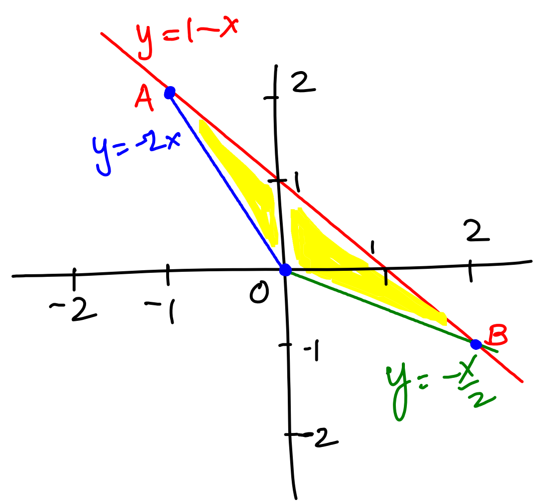
y varies from -2 to 1

— one with $-4 \le x \le -1$, and the other with $-1 \le x \le 0$.

Using horizontal cross sections, we can evaluate the integral in one step.

17. Sketch region of integration, then find area.

$$A = \int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-x/2}^{1-x} dy\,dx$$



1. $y=-2x$ to $y=1-x$
2. $y=-x/2$ to $y=1-x$

Points of intersection:

$-2x = 1-x \implies x=-1,\ y=2.$ $A(-1,2)$

$-\frac{x}{2} = 1-x \implies \frac{x}{2}=1$, i.e., $x=2, y=-1$.

$B(2,-1)$.

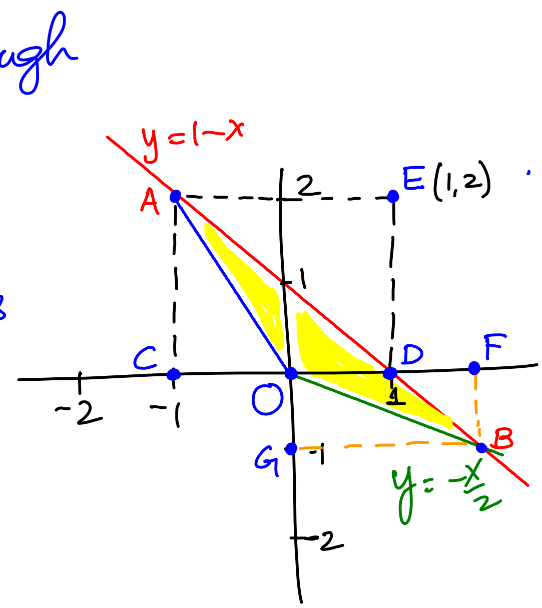
$$A = \int_{-1}^{0} y\Big|_{-2x}^{1-x} dx + \int_{0}^{2} y\Big|_{-x/2}^{1-x} dx$$

$$= \int_{-1}^{0} \left(1-x - -2x\right) dx + \int_{0}^{2} \left(1-x - -x/2\right) dx = \int_{-1}^{0} (1+x)\, dx + \int_{0}^{2} \left(1-\frac{x}{2}\right) dx$$

$$= x + \frac{x^2}{2}\Big|_{-1}^{0} + x - \frac{x^2}{4}\Big|_{0}^{2} = (0 - -1) + \frac{1}{2}\left(0 - (-1)^2\right) + (2-0) - \frac{1}{4}\left((2)^2 - (0)^2\right)$$

$$= 1 - \frac{1}{2} + 2 - 1 = \frac{3}{2}.$$

In this case, the region is simple enough for us to compute the area directly using geometric calculations – just to verify the result from integration. The total area is the sum of the areas of $\triangle OAD$ and $\triangle OBD$



Area of $\triangle OAD$ = Area of $\square ACDE$ − Area of $\triangle OAC$ − Area of $\triangle ADE$

$$= 2 \times 2 - \frac{1}{2}(1)(2) - \frac{1}{2}(2)(2) = 4 - 1 - 2 = 1.$$

Area of $\triangle OBD$ = Area of $\square OGBF$ − Area of $\triangle OBG$ − Area of $\triangle BDF$

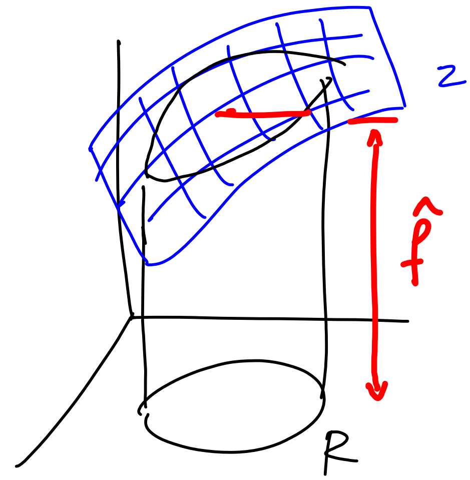
$$= 1 \times 2 - \frac{1}{2}(1)(2) - \frac{1}{2}(1)(1) = 2 - 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence the total area of the region $= 1 + \frac{1}{2} = \frac{3}{2}.$

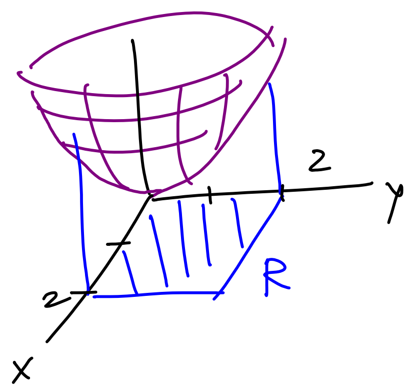
Average Value of f over R

$$\iint_R f\,dA = \text{Volume} = \text{Area}(R) \times \hat{f}$$

$$\Rightarrow \hat{f} = \frac{1}{\text{Area}(R)} \iint_R f\,dA$$



21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.



Area $= 2 \times 2 = 4$.

$$\hat{f} = \frac{1}{\text{Area}} \iint_R f(x,y)\,dA$$

$$= \frac{1}{4} \int_0^2 \int_0^2 (x^2+y^2)\,dy\,dx$$

$$= \frac{1}{4} \int_0^2 \left(x^2 y + \frac{1}{3}y^3 \Big|_0^2 \right) dx = \frac{1}{4} \int_0^2 \left(2x^2 + \frac{8}{3} \right) dx$$

$$= \frac{1}{4} \left(\frac{2}{3}x^3 + \frac{8}{3}x \right) \Big|_0^2 = \frac{1}{4} \left(\frac{2}{3}(2)^3 + \frac{8}{3}(2) \right) = \frac{8}{3}.$$

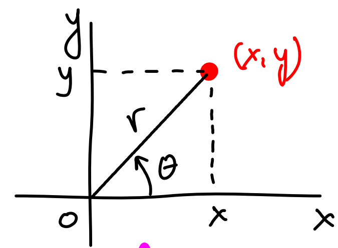
MATH 273 – Lecture 25 (11/20/2014)

Double Integrals in Polar Form (Section 14.4)

Recall: polar coordinates – r, θ

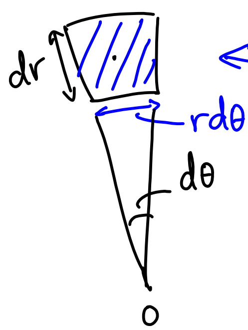
$$(x,y) \equiv (r\cos\theta, r\sin\theta)$$

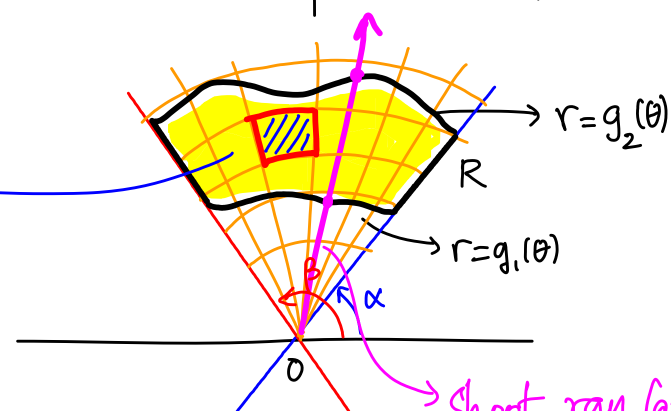
$$r = \sqrt{x^2+y^2}$$



$$\iint_R f(x,y)\,dA$$

dA → dx dy or dy dx





Shoot ray (arrow) out from origin through R – $g_1(\theta)$ and $g_2(\theta)$ are found at points of entry and exit from R.

In polar coordinates, $dA = (r\,d\theta)\cdot dr = r\,dr\,d\theta$

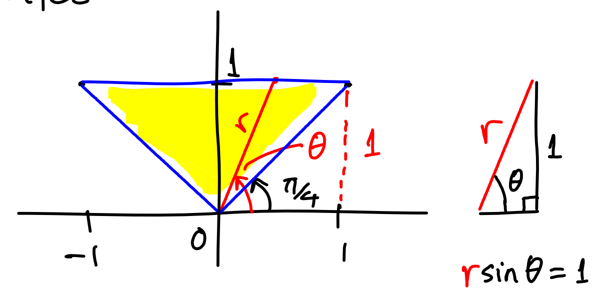
NOT $dr\,d\theta$!

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta.$$

3. Describe region R in polar coordinates.

$$\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$

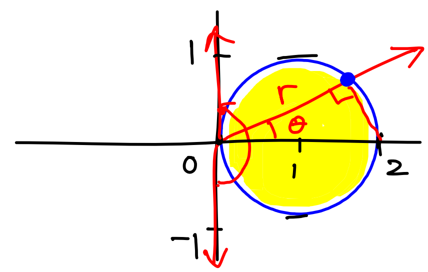
$$0 \le r \le \operatorname{cosec}\theta$$



Describe in polar coordinates

7. The region enclosed by the circle $x^2+y^2=2x$.

$$x^2+y^2=2x$$
$$x^2-2x+1+y^2=1$$
$$(x-1)^2+y^2=1$$



$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le 2\cos\theta.$$

plug in $x=r\cos\theta$, $y=r\sin\theta$ to get
$r^2=2r\cos\theta \Rightarrow r=2\cos\theta$

Notice that this computation is equivalent to the ray-shooting method. In the previous example, we can do a similar computation on $y=1$.

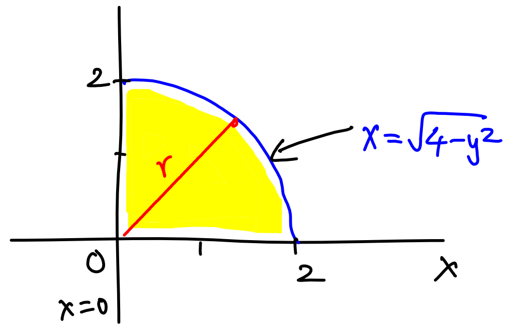
Finding limits of integration in polar coordinates

1. Sketch region.
2. Find r limits (shoot ray (arrow) from origin – find $r=g_1(\theta)$ where it enters R, and $r=g_2(\theta)$ where it leaves R).
3. Find θ limits.

11. Change the Cartesian integral to equivalent polar integral. Then evaluate the polar integral.

$$I = \int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$$

uses horizontal cross sections.



x : 0 to $\sqrt{4-y^2}$

$x=\sqrt{4-y^2} \iff x^2+y^2=4$

$0 \le \theta \le \pi/2$, $0 \le r \le 2$

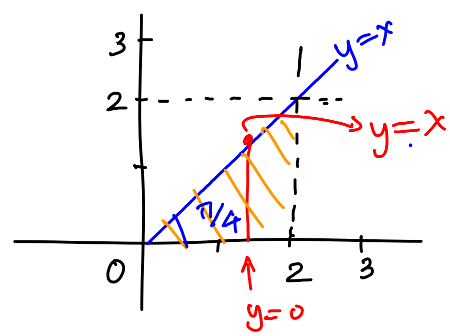
In polar form

$$I = \int_0^{\pi/2}\int_0^2 \underbrace{r^2}_{x^2+y^2} r\,dr\,d\theta = \int_0^{\pi/2}\left(\int_0^2 r^3 dr\right)d\theta = \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^2 d\theta$$

$$= \frac{16}{4}\int_0^{\pi/2} d\theta = 4.\ \theta\Big|_0^{\pi/2} = 4.\ \frac{\pi}{2} = 2\pi.$$

25. Sketch region R, and convert integral to Cartesian form.

$$I = \int_0^{\pi/4}\int_0^{2\sec\theta} r^5 \sin^2\theta\, dr\, d\theta$$



At $\theta=0$, $2\sec\theta = 2$

$\theta = \pi/4$ $2\sec\theta = \frac{2}{1/\sqrt{2}} = 2\sqrt{2}$

$0 \le x \le 2$

$0 \le y \le x$

$$I = \iint_R r^4 \sin^2\theta \underbrace{r\,dr\,d\theta}_{dy\,dx} = \iint_R \underbrace{r^2}_{(x^2+y^2)} \underbrace{r^2\sin^2\theta}_{y^2} r\,dr\,d\theta$$

$$= \iint_R (x^2+y^2)\, y^2\, dy\, dx = \int_0^2 \int_0^x (x^2+y^2)\, y^2\, dy\, dx.$$

Area in Polar Coordinates

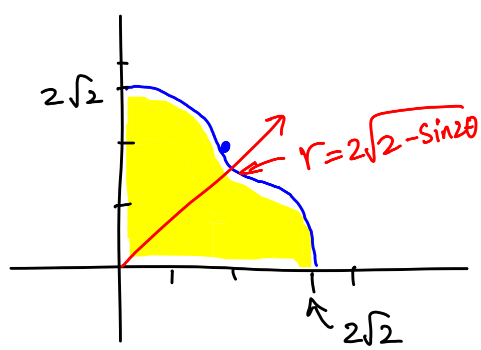
$$A = \iint_R 1 \cdot r\,dr\,d\theta = \iint_R r\,dr\,d\theta$$

29. Find area of region cut from the first quadrant by the curve $r = 2(2-\sin 2\theta)^{1/2} = 2\sqrt{2-\sin 2\theta}$.

$0 \le \theta \le \pi/2$

$r = 2\sqrt{2-\sin 2\theta}$

θ	r
0	$2\sqrt{2}$
$\pi/2$	$2\sqrt{2}$
$\pi/4$	2



$0 \le \theta \le \pi/2$

$0 \le r \le 2\sqrt{2-\sin 2\theta}$

$$A = \int_0^{\pi/2} \int_0^{2\sqrt{2-\sin 2\theta}} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} r^2 \Big|_0^{2\sqrt{2-\sin 2\theta}} \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 4(2-\sin 2\theta)\,d\theta \;=\; 4\theta \Big|_0^{\pi/2} + \cos 2\theta \Big|_0^{\pi/2}$$

$$= 4\left(\tfrac{\pi}{2} - 0\right) + \left(\cos\pi - \cos 0\right) = 2\pi + (-1-1)$$

$$= 2\pi - 2 = 2(\pi - 1).$$

We will not talk about triple integrals or polar coordinates in 3D due to time constraints. These topics extend the ideas we introduced in 2D to 3D.

MATH 273 – Lecture 26 (12/02/2014)

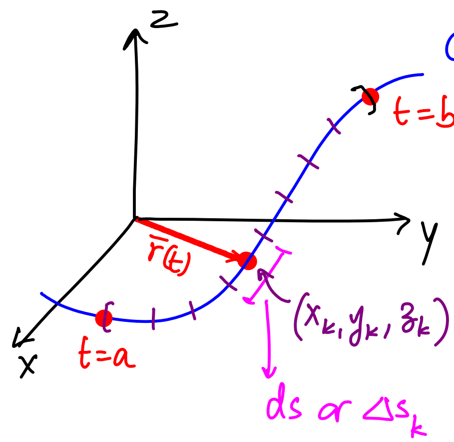
Integration in Vector fields (Chapter 15)

Rather than integrating over a region in $\mathbb{R}^2$ (2D space) or in $\mathbb{R}^3$ (3D space), we now integrate over a curve or over a surface.

We could use this concept to, for instance, calculate the work done in moving an object along a curved road (curve in 3D), or to find the mass of a curved metallic spring whose density is varying.

Today, we study line integrals.

Line Integrals



Def Let $f(x,y,z)$ be defined on a curve C given by

$$\bar{r}(t) = \underbrace{g(t)}_{x}\,\hat{i} + \underbrace{h(t)}_{y}\,\hat{j} + \underbrace{\ell(t)}_{z}\,\hat{k}, \quad a \le t \le b.$$

Then the line integral of f over C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k$$

same idea as used in single and double integrals – break region of integration R into small chunks, and evaluate the sum $\sum f(x_k, y_k, z_k)\,dA_k$, for instance.

How do we find ds?

With $\bar{v}(t) = \frac{d\bar{r}}{dt}$, we can write $ds = |\bar{v}(t)|dt$, since $\frac{ds}{dt} = |\bar{v}(t)|$

"velocity vector"

To evaluate $\int_C f(x,y,z)\,ds$, we

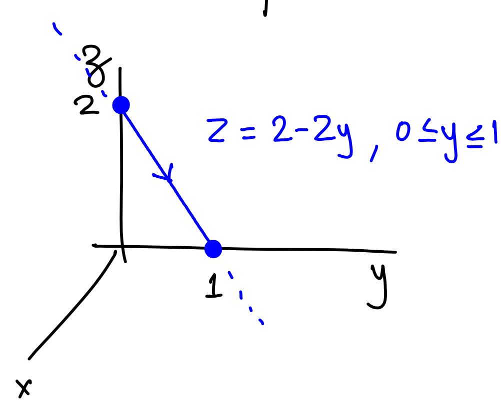
1. find smooth parametrization of C in the form $\bar{r}(t) = g(t)\hat{i} + h(t)\hat{j} + \ell(t)\hat{k}$, $a \le t \le b$, and

2. evaluate
$$\int_C f(x,y,z)\,ds = \int_a^b f\big(g(t), h(t), \ell(t)\big)\,|\bar{v}(t)|\,dt.$$

assume $|\bar{v}(t)| > 0$ over $a \le t \le b$

Probs 1-8

(b) Find parametric expression of the curve in picture.

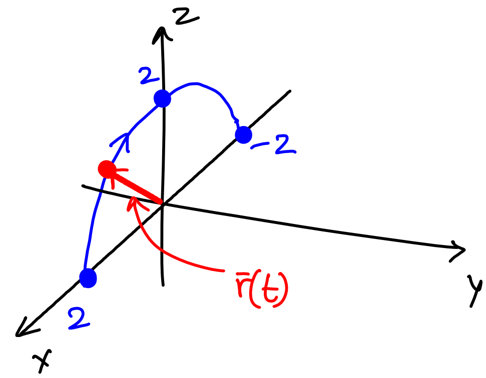


$y = t$, $0 \le t \le 1$ and
$z = 2-2t$. So, the curve is

$\bar{r}(t) = t\hat{j} + (2-2t)\hat{k}$, $0 \le t \le 1$.

given as Problem (6).

(h) $x = 2\cos t$, $0 \le t \le \pi$
$z = 2\sin t$,



The parametric curve is

$$\bar{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{k}, \quad 0 \le t \le \pi$$

9. Evaluate $\int_C (x+y)\,ds$ where C is the straight line segment $x = t$, $y = 1-t$, $z = 0$ from $(0,1,0)$ to $(1,0,0)$.

$a \le t \le b$

$$\bar{r}(t) = t\,\hat{i} + (1-t)\hat{j}, \quad 0 \le t \le 1.$$

$$f(x,y,z) = x+y = t + (1-t) = 1$$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = 1\cdot\hat{i} + (-1)\hat{j} = \hat{i} - \hat{j}. \quad \text{So } |\bar{v}(t)| = \sqrt{1^2+(-1)^2} = \sqrt{2}.$$

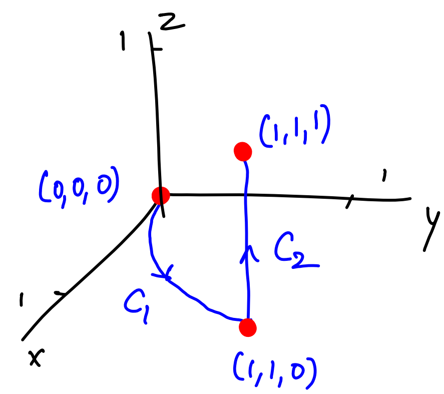
$$\text{So } \int_C f\,ds = \int_0^1 \underbrace{1}_{f}\cdot\underbrace{\sqrt{2}}_{|\bar{v}(t)|}\,dt = \sqrt{2}\,t\Big|_0^1 = \sqrt{2}.$$

In this problem, the parametric form is given to you. In some other problems, you have to find the parametric expression, and then evaluate the integral.

15. Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over path from $(0,0,0)$ to $(1,1,1)$ given by

$$C_1 : \bar{r}(t) = \underset{x}{t}\,\hat{i} + \underset{y}{t^2}\,\hat{j}, \quad 0 \le t \le 1,$$

$$C_2 : \bar{r}(t) = \underset{x}{\hat{i}} + \underset{y}{\hat{j}} + \underset{z}{t\hat{k}}, \quad 0 \le t \le 1.$$



$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds,$$ when C is the nonoverlapping union of C_1 and C_2.

C_1: $\bar{v}(t) = 1\hat{i} + 2t\hat{j} \qquad |\bar{v}(t)| = \sqrt{1+4t^2}$

$$\int_{C_1} f\,ds = \int_0^1 \big(\underset{x}{t} \underset{+\sqrt{y}}{+ \sqrt{t^2}} \underset{-z^2}{- 0^2}\big)\sqrt{1+4t^2}\,dt = \int_0^1 2t\sqrt{1+4t^2}\,dt$$

$$= \frac{1}{4}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^1 = \frac{1}{6}\left(5\sqrt{5}-1\right).$$

C_2: $\bar{v}(t) = \dfrac{d\bar{r}}{dt} = 0\hat{i} + 0\hat{j} + 1\hat{k} = \hat{k}$, So $|\bar{v}(t)| = 1$.

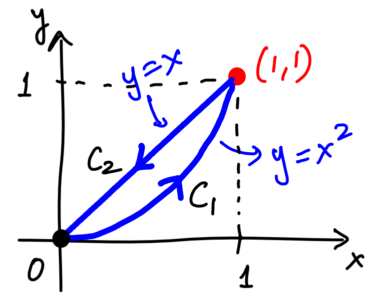
$$\int_{C_2} f\,ds = \int_0^1 \big(\underbrace{1 + \sqrt{1} - t^2}_{x+\sqrt{y}-z^2}\big)\,1.dt = \int_0^1 (2-t^2)\,dt = 2t - \frac{1}{3}t^3\Big|_0^1 = \frac{5}{3}.$$

So $$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \frac{1}{6}(5\sqrt{5}-1) + \frac{5}{3} = \frac{5\sqrt{5}}{6} + \frac{3}{2}.$$

MATH 273 – Lecture 27 (12/04/2014)

Line integrals, continued... (Section 15.1)

(25) Evaluate $\int_C (x+\sqrt{y})\,ds$ where C is given in the picture.



$C_1: y=x^2$ for $0\le x\le 1$

$$\bar{r}(t) = \underbrace{t}_{x}\hat{i} + \underbrace{t^2}_{y}\hat{j}, \quad 0\le t\le 1$$

$C_2: y=x$ for $1\le x\le 0$

$$\bar{r}(t) = \underbrace{(1-t)}_{x}\hat{i} + \underbrace{(1-t)}_{y}\hat{j}, \quad 0\le t\le 1$$

$$C_1: \bar{v}(t) = 1\cdot\hat{i} + 2t\hat{j} \implies |\bar{v}(t)| = \sqrt{1^2+(2t)^2} = \sqrt{1+4t^2}$$

$$f(x(t),y(t)) = x+\sqrt{y} = t+\sqrt{t^2} = 2t$$

$$\int_C f(x,y)\,ds = \int_0^1 f(t)\,|\bar{v}(t)|\,dt = \int_0^1 2t\sqrt{1+4t^2}\,dt = \frac{1}{4}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^1$$

$$= \frac{1}{6}\left(5\sqrt{5}-1\right).$$

$$C_2: \bar{v}(t) = -\hat{i}-\hat{j} \implies |\bar{v}(t)| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$$

$$\int_{C_2} f\,ds = \int_0^1 (\underbrace{1-t}_{x} + \underbrace{\sqrt{1-t}}_{\sqrt{y}})\sqrt{2}\,dt = \int_1^0 (u+\sqrt{u})\sqrt{2}(-du)$$

swap limits to cancel −ve sign

$u=1-t$
$du=-dt$

$$= \int_0^1 (u+\sqrt{u})\sqrt{2}\,du$$

$$= \sqrt{2}\left(\frac{1}{2}u^2 + \frac{2}{3}u^{3/2}\right)\Big|_0^1 = \sqrt{2}\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{7\sqrt{2}}{6}.$$

Hence $\displaystyle\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \frac{(5\sqrt{5}-1)}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5}+7\sqrt{2}-1}{6}$.

Line Integrals over Vector Fields

We now extend the idea of line integrals to the settings of a vector field. A gravitational field, or electromagnetic field are typical examples — we will study how to compute the work done in moving an object along a curve in such a field.

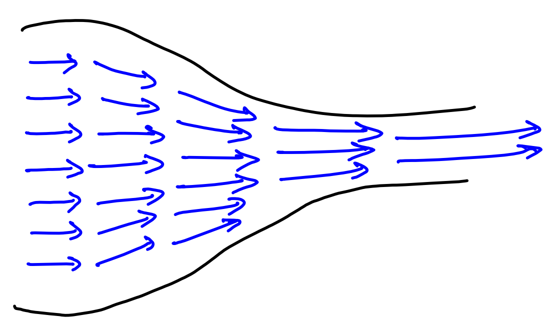
Def A **vector field** is a function that assigns a vector to each point in its domain.

$$F(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

F is continuous (differentiable) if $M, N,$ and P are continuous (differentiable).

Examples

1. fluid flow through a bottleneck



2. Gradient field:

$$\vec{F} = \nabla f$$

$$F(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

The vector assigned to each point is directed along the direction of largest rate of change of f, and its magnitude is the directional derivative along this direction.

Line Integral of a Vector Field

Curve C: $\bar{r}(t)$, $a \le t \le b$.

Unit tangent vector: $\hat{T} = \dfrac{\bar{v}(t)}{|\bar{v}(t)|} = \dfrac{d\bar{r}}{ds}$ → book uses $\mathbf{T}$ (bold T)

Def

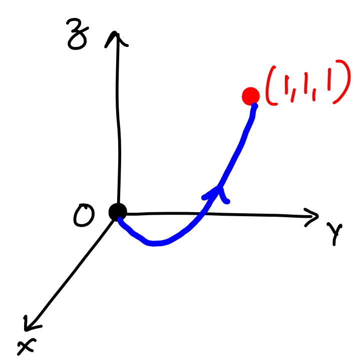
The line integral of $\bar{F}$ along C is $\displaystyle\int_C \bar{F}\cdot\hat{T}\,ds = \int_C \left(\bar{F}\cdot\frac{d\bar{r}}{ds}\right)ds = \int_C \bar{F}\cdot d\bar{r}$.

To evaluate, we compute $\displaystyle\int_a^b \left(\bar{F}(\bar{r}(t))\cdot\left(\frac{d\bar{r}}{dt}\right)\right)dt$.

7. $\bar{F} = 3y\,\hat{i} + 2x\,\hat{j} + 4z\,\hat{k}$

(b) C_2 : $\bar{r}(t) = \underbrace{t}_{x}\,\hat{i} + \underbrace{t^2}_{y}\,\hat{j} + \underbrace{t^4}_{z}\,\hat{k}$, $0 \le t \le 1$.

Find line integral of vector field F along C_2 from $(0,0,0)$ to $(1,1,1)$.



$\bar{F}(t) = 3\underbrace{t^2}_{y}\,\hat{i} + 2\underbrace{t}_{x}\,\hat{j} + 4\underbrace{t^4}_{z}\,\hat{k}$

$\dfrac{d\bar{r}}{dt} = 1\hat{i} + 2t\hat{j} + 4t^3\hat{k}$

$$\int_{C_2} \bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right)dt = \int_0^1 \left(3t^2 + 2t\times 2t + 4t^4\times 4t^3\right)dt = \int_0^1 \left(7t^2 + 16t^7\right)dt$$

$$= \frac{7}{3}t^3 + \frac{16}{8}t^8 \Big|_0^1 = \frac{7}{3} + 2 = \frac{13}{3}.$$

This line integral can be used to evaluate the work done in moving an object along curve C from $\underbrace{t=a}_{\text{pt. A}}$ to $\underbrace{t=b}_{\text{pt B}}$ over a force field $\bar{F}$.

$$W = \int_C \bar{F} \cdot \hat{T}\, ds = \int_a^b \left(\bar{F}(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\right) dt.$$

(19). $\bar{F} = xy\,\hat{i} + y\,\hat{j} - yz\,\hat{k}$, $\bar{r}(t) = \underbrace{t}_{x}\,\hat{i} + \underbrace{t^2}_{y}\,\hat{j} + \underbrace{t}_{z}\,\hat{k}$, $0 \le t \le 1$.

Find the work done in moving along C in the direction of increasing t (from $t=0$ to $t=1$).

$$\bar{F}(\bar{r}(t)) = t\,t^2\,\hat{i} + t^2\,\hat{j} - t^2 t\,\hat{k} = t^3\,\hat{i} + t^2\,\hat{j} - t^3\,\hat{k}$$

$$\frac{d\bar{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}. \qquad \bar{F} \cdot \left(\frac{d\bar{r}}{dt}\right) = t^3 + 2t^3 - t^3 = 2t^3$$

$$W = \int_0^1 \bar{F} \cdot \left(\frac{d\bar{r}}{dt}\right) dt = \int_0^1 2t^3\, dt = \frac{2}{4} t^4 \Big|_0^1 = \frac{1}{2}.$$

MATH 273 – Lecture 28 (12/09/2014)

Flow and Circulation

If $\bar{F}$ is a continuous velocity field, then the **flow** along the curve C from $A = \bar{r}(a)$ to $B = \bar{r}(b)$ is given by

$$\text{Flow} = \int_C F \cdot \hat{T}\, ds. \longrightarrow$$

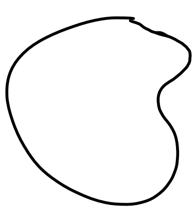
evaluate it similar to how we compute work.

If $A = B$, C is a closed curve, and the flow is then called the **circulation** around C.

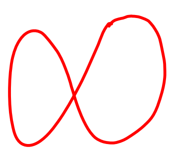
We will consider, in detail, **simple closed** curves

simple: does not cross itself

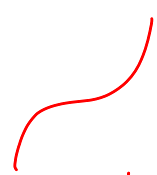
closed: loop



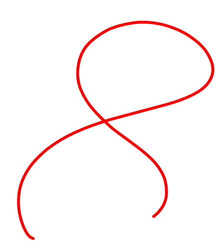
simple, closed



closed, not simple



simple, not closed

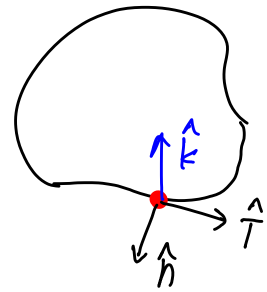


not simple
not closed

Circulation – adds $\bar{F}\cdot\hat{T}$ over C

We now consider $\bar{F}\cdot\hat{n}$, where $\hat{n}$ is the unit normal vector.

$\hat{k}$, the z-unit vector, points up from the plane, and is perpendicular to both $\hat{T}$ and $\hat{n}$.



Def If C is a smooth simple closed curve in the domain of a vector field $\bar{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ in the plane, and $\hat{n}$ is the outward pointing normal vector on C, then the

$$\textbf{Flux across } C = \int_C \bar{F}\cdot\hat{n}\,ds.$$

How do we compute $\hat{n}$? We orient C in the counter-clockwise (CCW) direction. Then $\hat{n} = \hat{T}\times\hat{k}$ (right-hand rule for cross-product). So

$$\hat{n} = \underbrace{\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right)}_{\hat{T}}\times\hat{k} = \underbrace{\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}}_{\hat{n}}.$$

So $\bar{F}\cdot\hat{n} = (M\hat{i}+N\hat{j})\cdot\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right)$

$$= M\frac{dy}{ds} - N\frac{dx}{ds}.$$

Thus, flux across $C = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds$

i.e., $$C = \int_C M\,dy - N\,dx.$$

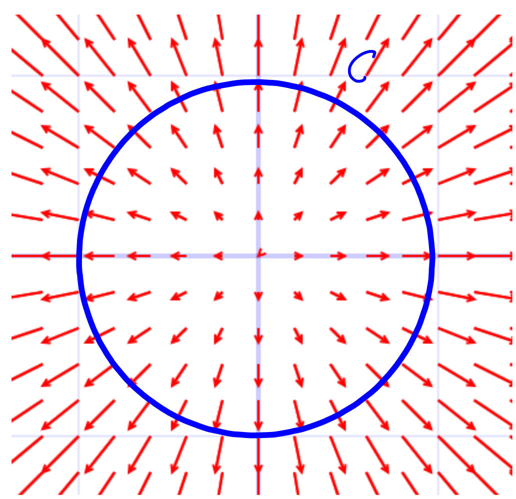
Prob 29. $\bar{F} = x\hat{i}+y\hat{j}$. (a) $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0\le t\le 2\pi$.

C (unit circle)

Find circulation of $\bar{F}$ around C, and flux of $\bar{F}$ across C.

$\bar{r}(t) = \underbrace{\cos t}_{x}\,\hat{i} + \underbrace{\sin t}_{y}\,\hat{j}$

$\bar{F} = \underbrace{x}_{M}\,\hat{i} + \underbrace{y}_{N}\,\hat{j}$



$x = \cos t$ $\qquad$ $y = \sin t$

$dx = -\sin t\,dt$ $\qquad$ $dy = \cos t\,dt$

$$\text{Circulation} = \int_C \bar{F}\cdot\hat{T}\,ds = \int_0^{2\pi} \bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right)dt$$

$$\frac{d\bar{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j}, \quad \bar{F} = \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$\bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right) = -\cos t\sin t + \cos t\sin t = 0.$$ Hence circulation $= 0$.

Flux: $\hat{n} = \hat{T}\times\hat{k} = \left(\frac{d\bar{r}}{dt}\right)\times\hat{k} = \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right)\times\hat{k}$

$$= \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$\text{Flux} = \int_C \bar{F}\cdot\hat{n}\,ds = \int_0^{2\pi} (\underbrace{\cos^2 t + \sin^2 t}_{1})\,dt = 2\pi.$$

Alternatively, $F = M\hat{i} + N\hat{j}$ where $M = x$, $N = y$

$$\text{Flux} = \int_C M\,dy - N\,dx = \int_0^{2\pi} \underbrace{\cos t}_{M}\cdot\underbrace{\cos t\,dt}_{dy} - \underbrace{\sin t}_{N}\underbrace{(-\sin t\,dt)}_{dx}$$

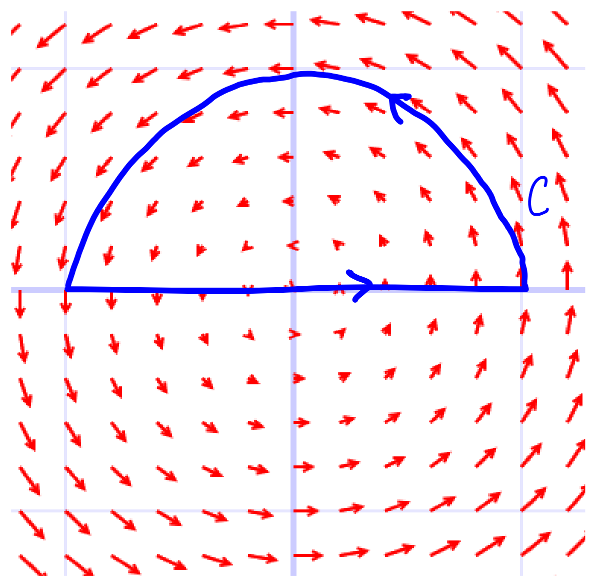
$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi.$$

33. $\bar{F} = -y\hat{i} + x\hat{j}$

C: closed semicircular arch $\bar{r}_1(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$, $0 \le t \le \pi$

followed by the line segment $\bar{r}_2(t) = t\hat{i}$, $-a \le t \le a$.

Flow_1: $\bar{r}_1(t) = \underbrace{a\cos t}_{x}\,\hat{i} + \underbrace{a\sin t}_{y}\,\hat{j}$ $dx = -a\sin t\,dt$, $dy = a\cos t\,dt$



$$\bar{F}_1 = \underbrace{-a\sin t}_{M_1}\,\hat{i} + \underbrace{a\cos t}_{N_1}\,\hat{j}$$

$$\frac{d\bar{r}_1}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

$$F_1 \cdot \left(\frac{d\bar{r}_1}{dt}\right) = a^2\sin^2 t + a^2\cos^2 t = a^2$$

$$\text{Flow}_1 = \int_0^{\pi} \bar{F}_1 \cdot \left(\frac{dr}{dt}\right) dt = \int_0^{\pi} a^2\,dt = \pi a^2.$$

Flow_2: $\bar{r}_2(t) = \underbrace{t}_{x}\,\hat{i}$, $-a \le t \le a$ $y=0$

$$\frac{d\bar{r}_2}{dt} = \hat{i}$$

$$\bar{F}_2 = 0\hat{i} + t\hat{j}$$

$$\bar{F}_2 \cdot \left(\frac{d\bar{r}_2}{dt}\right) = t\hat{j} \cdot \hat{i} = 0.$$

So $\text{Flow}_2 = 0.$

Circulation around C = $\text{Flow}_1 + \text{Flow}_2 = \pi a^2$.

$$\underline{\text{Flux}_1} = \int_{C_1} M_1\,dy - N_1\,dx$$

$$= \int_0^{\pi} \underbrace{-a\sin t}_{M_1}\ \underbrace{a\cos t\,dt}_{dy} - \underbrace{(a\cos t)}_{N_1}\underbrace{(-a\sin t\,dt)}_{dx} = 0.$$

$\underline{\text{Flux}_2}$ $\quad \bar{r}_2(t) = \underbrace{t}_{x}\,\hat{i} \qquad y=0 \qquad dx = dt$

$\bar{F}_2 = \underbrace{t}_{N_2}\,\hat{j} \qquad M_2 = 0 \qquad dy = 0\,dt = 0$

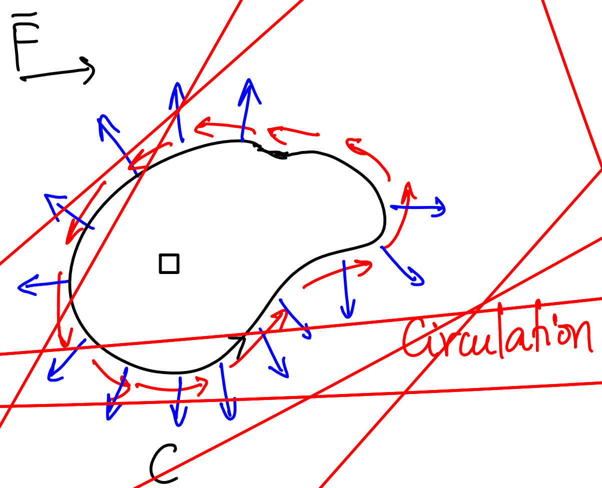
$$\text{Flux}_2 = \int_{-a}^{a} M_2\,dy - N_2\,dx = \int_{-a}^{a} 0\cdot 0 - t\,dt = \int_{-a}^{a} -t\,dt$$

$$= -\frac{1}{2}t^2\Big|_{-a}^{a} = -\frac{1}{2}\left(a^2 - (-a)^2\right) = 0.$$

$$\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0.$$

MATH 273 – Lecture 29 (12/11/2014)

Vector field $\bar{F}$ and a

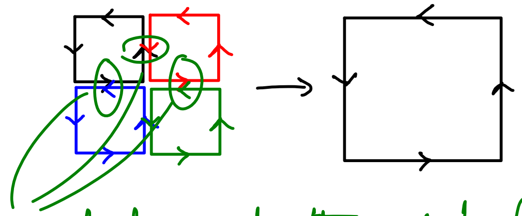


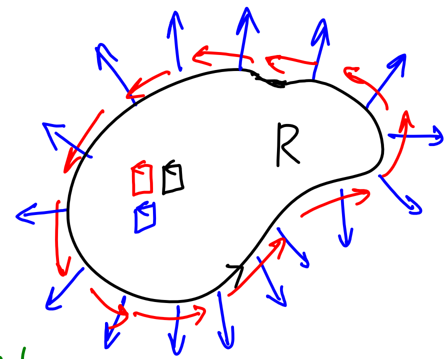
Flux across C $= \oint_C \bar{F} \cdot \hat{n}\, ds$.

Circulation around C $= \oint_C \bar{F} \cdot \hat{T}\, ds$.

We have seen previously, $\iint_R f(x,y)\, dA$.

Idea: Consider flux across the boundary of small boxes – $\Delta x\, \Delta y$, What about circulation?





circulations at the interface cancel!

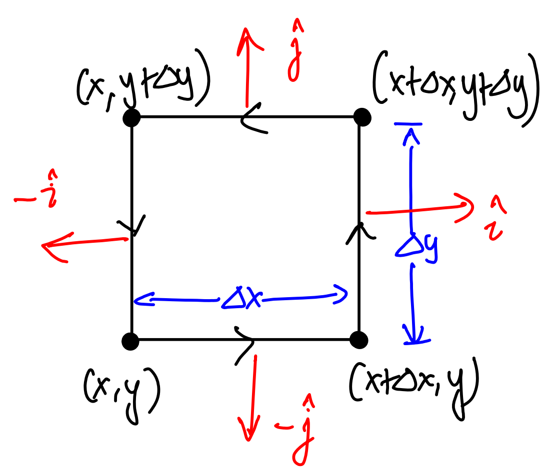
It does seem to work out, in both cases! Green's theorem formalizes this idea. It defines circulation and flux densities at (x,y), which when integrated over R give the circulation and flux.

Green's theorem in the Plane (Section 15.4)

The vector field is $\bar{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$.

Def The **divergence** or **flux density** of the vector field $\bar{F} = M\hat{i} + N\hat{j}$ at (x,y) is

$$\text{div}\, F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$



bottom : $\bar{F}\cdot(-\hat{j})\,\Delta x = -N\Delta x$

$= -N(x,y)\Delta x$

top : $\bar{F}\cdot(\hat{j})\,\Delta x = N(x, y+\Delta y)\Delta x$

$$\text{flux across top + bottom} = \underbrace{\Big(N(x,y+\Delta y) - N(x,y)\Big)}_{\frac{\partial N}{\partial y}\cdot\Delta y}\Delta x$$

$$\frac{\partial N}{\partial y}\cdot \Delta y\ \Delta x$$

Similarly, we can get that

$$\text{flux across left + right} = \frac{\partial M}{\partial x}\Delta x\,\Delta y.$$

$$\text{So, total flux} = \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\text{divergence}}\Delta x \Delta y$$

$$\Rightarrow \text{flux density} = \frac{\text{total flux}}{\Delta A} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Green's theorem (flux-divergence or Normal form)

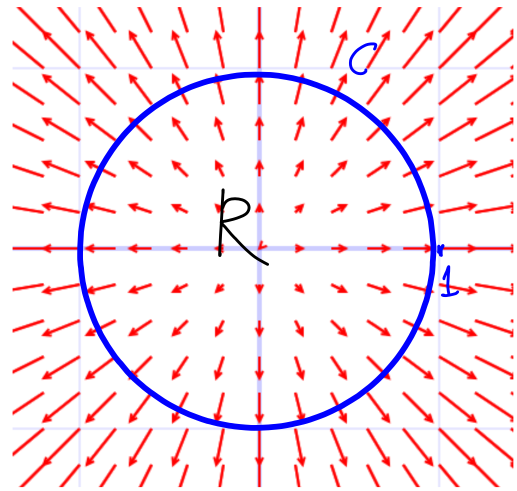
Let C be a piecewise smooth closed curve enclosing region R in the plane. Let $\bar{F} = M\hat{i} + N\hat{j}$, with M, N having continuous first partial derivatives in a open region containing C. Then the outward flux of $\bar{F}$ across C is equal to the double integral of $\operatorname{div}\bar{F}$ over R.

$$\oint_C \bar{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\operatorname{div}\bar{F}} dx\,dy.$$

Back to Problem 29 (Section 15.2)

$\bar{F} = \underset{M}{x}\hat{i} + \underset{N}{y}\hat{j}$. (a) $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$.

C (unit circle)



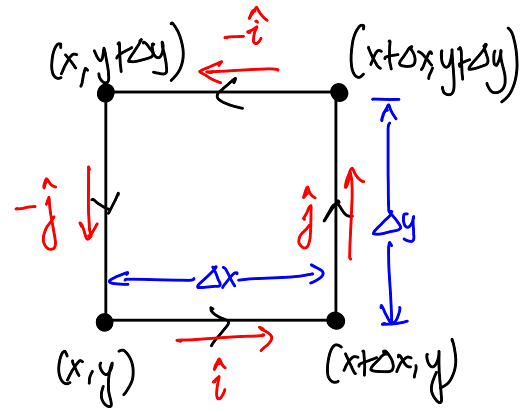
We obtained the flux across $C = 2\pi$.

$$\operatorname{div}\bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \qquad M = x,\ N = y$$

$$= 1 + 1 = 2.$$

$$\oint_C \bar{F}\cdot\hat{n}\,ds = \iint_R 2\,dA = 2(\text{Area}) = 2\pi(1)^2 = 2\pi.$$

Circulation Density



bottom : $\bar{F}\cdot\hat{i}\,\Delta x = M(x,y)\Delta x$

top : $\bar{F}\cdot(-\hat{i})\Delta x = -M(x, y+\Delta y)\Delta x$

top + bottom : $-\big(M(x, y+\Delta y) - M(x,y)\big)\Delta x$

$$\text{circulation} = -\frac{\partial M}{\partial y}\Delta y\,\Delta x$$

Similarly, for left + right, we get

$$\text{circulation} = \frac{\partial N}{\partial x}\Delta x \Delta y.$$

$$\text{So, total circulation} = \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}}\underbrace{\Delta x \Delta y}_{\text{area}}$$

$$\text{circulation density} = \frac{\text{circulation}}{\text{area}}$$

$$\Rightarrow \textbf{Circulation density} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

also called the k-component of the **curl**, $\text{curl}\,\bar{F}\cdot\hat{k}$.

Green's theorem (tangential form)

The CCW (counterclockwise) circulation of $\bar{F} = M\hat{i} + N\hat{j}$ around C is equal to the double integral of the circulation density of $\bar{F}$ over R.

$$\oint_C \bar{F}\cdot\hat{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

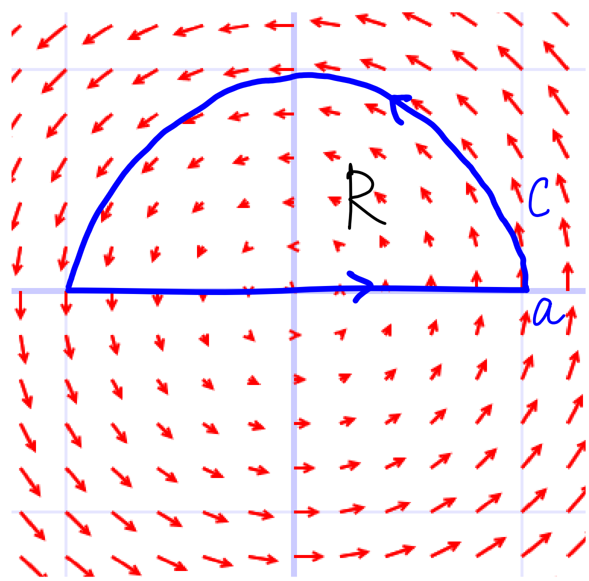
Back to problem 33. (Section 15.2)

$$\bar{F} = \underbrace{-y}_{M}\,\hat{i} + \underbrace{x}_{N}\,\hat{j}$$

C : closed semicircular arch $\bar{r}_1(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$, $0 \le t \le \pi$

followed by the line segment $\bar{r}_2(t) = t\,\hat{i}$, $-a \le t \le a$.

We had obtained the circulation around $C = \pi a^2$.



$M = -y$, $N = x$

$$\frac{\partial N}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1.$$

Using Green's formula,

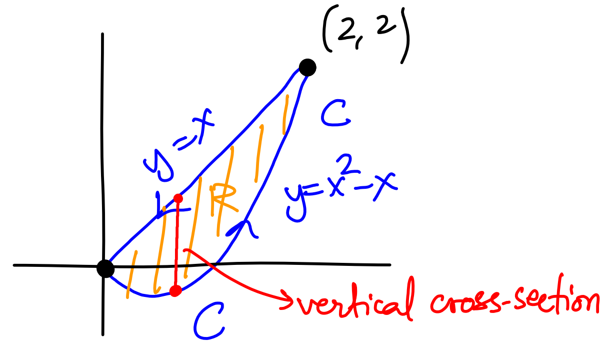
$$\oint_C \bar{F}\cdot\hat{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (1 - -1)\,dA = 2\iint_R dA = 2\,(\text{Area})$$

$$= 2\left(\frac{1}{2}\pi a^2\right) = \pi a^2.$$

Prob 11, Section 15.4

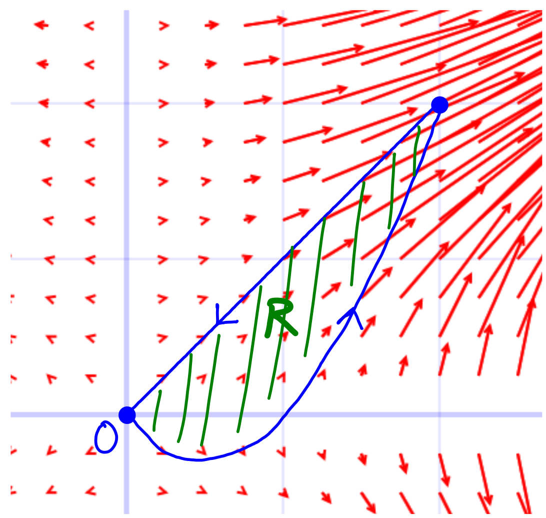
$$\bar{F} = \underbrace{x^3y^2}_{M}\,\hat{\imath} + \underbrace{\tfrac{1}{2}x^4y}_{N}\,\hat{\jmath}$$



Find circulation around and flux across C using Green's theorem.

$$M = x^3y^2 \Rightarrow \frac{\partial M}{\partial x} = 3x^2y^2\ ;\quad \frac{\partial M}{\partial y} = 2x^3y$$

$$N = \tfrac{1}{2}x^4y \Rightarrow \frac{\partial N}{\partial x} = 2x^3y\ ;\quad \frac{\partial N}{\partial y} = \tfrac{1}{2}x^4$$



Circulation: $\displaystyle\oint_C \bar{F}\cdot\hat{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$

$$= \iint_R (2x^3y - 2x^3y)\,dA = 0.$$

Flux: $\displaystyle\oint_C \bar{F}\cdot\hat{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dA$

$$= \int_0^2\int_{x^2-x}^{x}\left(3x^2y^2 + \tfrac{1}{2}x^4\right)dy\,dx = \int_0^2 \left(x^2y^3 + \tfrac{1}{2}x^4y \Big|_{\underbrace{x^2-x}_{x(x-1)}}^{x}\right)dx$$

$$= \int_0^2 \left[x^2\left(x^3 - x^3\underbrace{(x-1)^3}_{x^3-3x^2+3x-1}\right) + \tfrac{1}{2}x^4\underbrace{\left(x - x(x-1)\right)}_{2x-x^2}\right]dx$$

$$= \int_0^2 \left[x^2\left(x^3 - x^6 + 3x^5 - 3x^4 + x^3\right) + x^5 - \tfrac{1}{2}x^6\right]dx = \int_0^2 \left[3x^5 - \tfrac{7}{2}x^6 + 3x^7 - x^8\right]dx$$

$$= \tfrac{1}{2}x^6 - \tfrac{1}{2}x^7 + \tfrac{3}{8}x^8 - \frac{x^9}{9}\Big|_0^2 = \underbrace{\frac{(2)^6}{2}}_{32} - \underbrace{\frac{(2)^7}{2}}_{64} + \underbrace{\frac{3}{8}(2)^8}_{3\times 32} - \frac{(2)^9}{9} = 64 - \frac{512}{9} = \frac{64}{9}.$$

MATH 273 – Lecture 30 (12/14/2014)

Review for the final exam

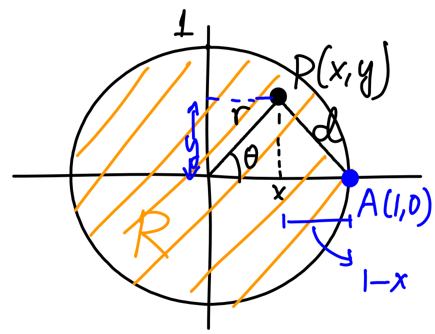
Recall, problems in the final will be from sections covered after Exam II – 14.14.4, 15.1, 15.2, & Green's theorem evaluation.

5. **(10)** Similar to the definition given in Cartesian (x, y) coordinates, the average value of the function $f(r, \theta)$ over a region R in polar coordinates is given by

$$\hat{f} = \frac{1}{\text{Area}(R)} \iint_R f(r, \theta)\, r dr d\theta.$$

Using the above definition, find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \leq 1$ to the boundary $A(1, 0)$.

$$d^2 = (1-x)^2 + y^2$$
$$= (1 - r\cos\theta)^2 + (r\sin\theta)^2$$
$$= 1 - 2r\cos\theta + \underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{r^2}$$
$$= 1 - 2r\cos\theta + r^2$$



$\text{Area}(R) = \pi(1)^2 = \pi$

$$\text{Average squared distance} = \frac{1}{\text{Area}(R)} \iint_R (1 - 2r\cos\theta + r^2)\, r dr d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (1 - 2r\cos\theta + r^2)\, r dr d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{2}{3} r^3 \cos\theta + \frac{r^4}{4} \Big|_0^1 \right) d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\underbrace{\frac{1}{2} - \frac{2}{3}\cos\theta + \frac{1}{4}}_{\frac{3}{4}} \right) d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3}{4} - \frac{2}{3}\cos\theta \right) d\theta = \frac{1}{\pi} \left[\frac{3}{4}\theta - \frac{2}{3}\sin\theta \right] \Big|_0^{2\pi} = \frac{1}{\cancel{\pi}} \frac{3}{\cancel{4}_2} (\cancel{2\pi} - 0) - \frac{2}{3}(0 - 0)$$

$$= \frac{3}{2}.$$

30·2

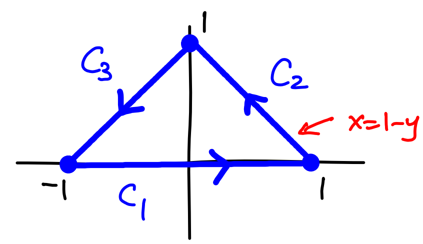
9. **(20)** Find the flux and circulation by evaluating the line integrals (in Part 9a). Then compute these quantities using Green's theorem (in Part 9b).

(a) Find the flux across and the circulation around the triangle with vertices $(1, 0)$, $(0, 1)$, and $(-1, 0)$ of the vector field $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$. Recall that you want the *outward* flux and the *counterclockwise* circulation.

(b) For $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ and a piecewise smooth closed curve C which bounds the region R, two forms of Green's theorem (in 2D) can be specified as follows. Here, $\mathbf{T}$ is the unit tangent and $\mathbf{n}$ is the unit normal vector at each point on C.

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \quad \text{and} \quad \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Find the circulation and flux for the field $\mathbf{F}$ and closed curve C given in Part 9a by evaluating the corresponding double integrals specified by Green's theorem.



$C_1: \ \bar{r}_1(t) = \underbrace{t}_{x}\,\hat{i}, \quad -1 \le t \le 1.$

$\hat{T}_1 = \dfrac{d\bar{r}_1}{dt} = \hat{i}, \quad \hat{n}_1 = \hat{T}_1 \times \hat{k} = \hat{i} \times \hat{k} = -\hat{j}.$

$\bar{F}_1(t) = \underbrace{(t+0)}_{x+y}\hat{i} \underbrace{-}_{-} \underbrace{(t^2+0^2)}_{x^2+y^2}\hat{j} = t\hat{i} - t^2\hat{j}$

$$\text{Circulation}_1 = \int_{C_1} \bar{F}_1\cdot\hat{T}_1\,ds = \int_{-1}^{1} t\,dt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0.$$

$$\text{Flux}_1 = \int_{C_1} \bar{F}_1\cdot\hat{n}_1\,ds = \int_{-1}^{1} t^2\,dt = \frac{1}{3}t^3\Big|_{-1}^{1} = \frac{1}{3}\left(1^3 - (-1)^3\right) = \frac{2}{3}.$$

$C_2: \ x = 1-y, \ 0 \le y \le 1$ gives $\bar{r}_2(t) = \underbrace{(1-t)}_{x}\,\hat{i} + \underbrace{t}_{y}\,\hat{j}, \quad 0 \le t \le 1.$

$\hat{T}_2 = \dfrac{d\bar{r}_2}{dt} = -\hat{i} + \hat{j}; \quad \hat{n}_2 = \hat{T}_2 \times \hat{k} = \hat{j} + \hat{i} = \hat{i} + \hat{j}.$

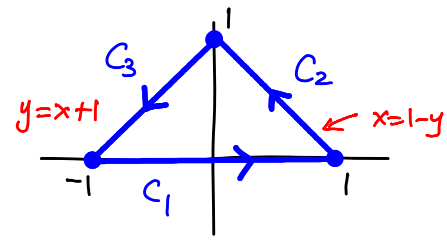
$\bar{F}_2 = \underbrace{(1-t+t)}_{x+y}\hat{i} \underbrace{-}_{-} \underbrace{((1-t)^2 + t^2)}_{x^2+y^2}\hat{j} = \hat{i} - (2t^2 - 2t + 1)\hat{j}.$

$$\text{Circulation}_2 = \int_{C_2} \bar{F}_2\cdot\hat{T}_2\,ds = \int_0^1 \Big[-1 \underbrace{-(2t^2-2t+1)}_{-2t^2+2t-2}\Big]dt = -\frac{2}{3}t^3 + t^2 - 2t\Big|_0^1$$

$$= -\frac{2}{3} + 1 - 2 = -\frac{5}{3}.$$

$$\text{Flux}_2 = \int_{C_2} \bar{F}_2 \cdot \hat{n}_2 \, ds = \int_0^1 [\underbrace{1-(2t^2-2t+1)}_{-2t^2+2t}] \, dt = -\frac{2}{3}t^3 + t^2 \Big|_0^1 = -\frac{2}{3}+1 = \frac{1}{3}.$$

C_3: $\bar{r}_3(t) = \underbrace{-t}_{x}\,\hat{i} + \underbrace{(1-t)}_{y}\,\hat{j}$, $0 \le t \le 1$.



Could also try $-1 \le t \le 0$, with $\bar{r}_3(t) = (-t-1)\hat{i} + -t\hat{j}$.

$$\hat{T}_3 = \frac{d\bar{r}_3}{dt} = -\hat{i}-\hat{j} \; ; \; \hat{n}_3 = \hat{T}_3 \times \hat{k} = \hat{j}-\hat{i} = -\hat{i}+\hat{j}.$$

$$\bar{F}_3(t) = (\underbrace{-t+(1-t)}_{x+y})\hat{i} - (\underbrace{(-t)^2 + (1-t)^2}_{x^2+y^2})\hat{j} = (1-2t)\hat{i} - (2t^2-2t+1)\hat{j}.$$

$$\text{Circulation}_3 = \int_{C_3} \bar{F}_3 \cdot \hat{T}_3 \, ds = \int_0^1 [-(\cancel{1}-\cancel{2t}) + (2t^2 - \cancel{2t} + \cancel{1})] \, dt = \int_0^1 2t^2 \, dt = \frac{2}{3}t^3 \Big|_0^1 = \frac{2}{3}.$$

$$\text{Flux}_3 = \int_{C_3} \bar{F}_3 \cdot \hat{n}_3 \, ds = \int_0^1 [\underbrace{-(1-2t)-(2t^2-2t+1)}_{-2t^2+4t-2}] \, dt = -\frac{2}{3}t^3 + 2t^2 - 2t \Big|_0^1 = -\frac{2}{3}+2-2 = -\frac{2}{3}.$$

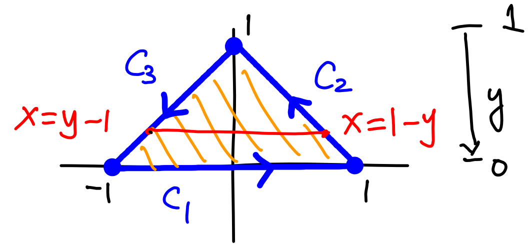
$$\text{Total Circulation} = \text{Circulation}_1 + \text{Circulation}_2 + \text{Circulation}_3$$
$$= 0 - \frac{5}{3} + \frac{2}{3} = -1.$$

$$\text{Total Flux} = \frac{2}{3} + \frac{1}{3} - \frac{2}{3} = \frac{1}{3}.$$

(b) $\bar{F} = \underbrace{(x+y)}_{M}\hat{i} - \underbrace{(x^2+y^2)}_{N}\hat{j}$ So $M = x+y$, $N = -(x^2+y^2)$.

$$\frac{\partial M}{\partial x} = 1; \quad \frac{\partial M}{\partial y} = 1; \quad \frac{\partial N}{\partial x} = -2x, \quad \frac{\partial N}{\partial y} = -2y$$

$$\text{Circulation} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$



$$= \int_0^1 \int_{y-1}^{1-y} (-2x-1)\,dx\,dy = \int_0^1 \left(-x^2 - x \Big|_{y-1}^{1-y}\right) dy$$

$$= -\int_0^1 \left[\cancel{(1-y)^2} + (1-y) - \cancel{(y-1)^2} - (y-1)\right] = -\int_0^1 (2-2y)\,dy = 2y - y^2 \Big|_1^0$$

$$= 2(0-1) - (0^2 - (1)^2) = -2 + 1 = -1.$$

$$\text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \int_0^1 \int_{y-1}^{1-y} (1-2y)\,dx\,dy = \int_0^1 \left((1-2y)\,x \Big|_{y-1}^{1-y}\right) dy$$

$$= \int_0^1 \underbrace{(1-2y)\left[1-y-(y-1)\right]}_{(1-2y)(2-2y)} dy = \int_0^1 (4y^2 - 6y + 2)\,dy = \frac{4}{3}y^3 - 3y^2 + 2y \Big|_0^1$$

$$= \frac{4}{3} - 3 + 2 = \frac{1}{3}.$$

30.5

10. **(12)** Decide whether each of the following statements is *True* or *False*. **Justify** your answer.

(a) The value of a line integral of a function along a curve depends only on the values of the function at the end points.

(b) Integration using polar coordinates could be applied only to regions that are circular in shape and bounded by smooth curves.

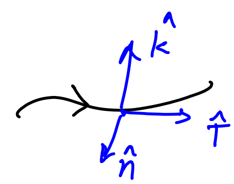
(c) The flux across the region bounded by a simple closed curve changes sign when the direction of traversal of the curve is reversed from counterclockwise to clockwise.

(d) Values of the circulation and the flux of a vector field **F** around and across the region R bounded by a smooth closed curve C can never be equal.

(a) FALSE. The value depends typically on the path taken between the end points.

(b) FALSE. Polar coordinates are an equivalent alternative coordinate system. Its use does not depend on the actual function or region in the integration.

(c) TRUE. The unit normal is reversed at each point by reversing the orientation of the curve.



(d) FALSE. They could be equal. Trivially, consider $\bar{F} = 0\hat{\imath} + 0\hat{\jmath}$, the zero vector field.