MATH 273 – Lecture 10 (09/25/2014)

Sections covered up to end of today's lecture will be relevant for Exam 1, on Thursday Oct 2.

Hw 5 is due next Tuesday by 5:00 pm.

Tangent plane and normal line to surface $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$

Tangent plane: $(f_x)_{P_0}(x-x_0)+(f_y)_{P_0}(y-y_0)+(f_z)_{P_0}(z-z_0)=0.$

normal line: $x=x_0+(f_x)_{P_0}t,\ y=y_0+(f_y)_{P_0}t,\ z=z_0+(f_z)_{P_0}t,\ -\infty<t<\infty.$

Prob 5 Find tangent plane and normal line at $P_0(0,1,2)$ to surface $\underbrace{\cos\pi x - x^2y + e^{xz} + yz = 4}_{f(x,y,z)=c}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k} = \left(-\pi\sin(\pi x)-2xy+ze^{xz}+0\right)\hat{i} + \left(0-x^2+0+z\right)\hat{j}+\left(0-0+xe^{xz}+y\right)\hat{k}$$

At $P_0(\underset{x_0,y_0,z_0}{0,1,2})$, $(\nabla f)_{P_0} = (0-0+2e^0)\hat{i}+(-0^2+2)\hat{j}+(0\cdot e^0+1)\hat{k}$

$$= 2\hat{i}+2\hat{j}+\hat{k}.$$

So, tangent plane is $2(x-\underset{x_0}{0})+2(y-\underset{y_0}{1})+1(z-\underset{z_0}{2})=0$

i.e, $2x+2y+z-4=0$

Normal line is given by: $x=0+2t,\ y=1+2t,\ z=2+1t,\ -\infty<t<\infty,$

i.e., $x=2t,\ y=1+2t,\ z=2+t.$

We could also specify the surface in the form $z = f(x,y)$, instead of $f(x,y,z) = c$.

<u>Prob 9</u> $z = \ln(x^2+y^2)$, $P_0(1,0,0)$. Find the tangent plane.

$z = \ln(x^2+y^2)$ can be written as $\underbrace{\ln(x^2+y^2) - z = 0}_{f(x,y,z)=c}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \frac{2x}{(x^2+y^2)}\hat{i} + \frac{2y}{(x^2+y^2)}\hat{j} - 1\hat{k}$$

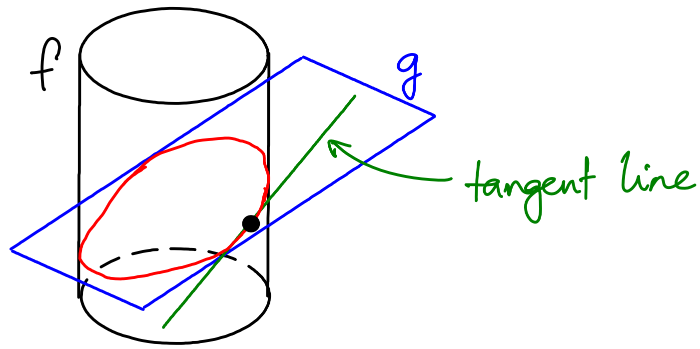
$$(\nabla f)_{P_0} = \frac{2}{1^2}\hat{i} + \frac{0}{1^2}\hat{j} - \hat{k} = 2\hat{i} - \hat{k}. \qquad (x_0, y_0, z_0) = (1,0,0)$$

So the tangent plane is $2(x-1) + 0(y-0) + (-1)(z-0) = 0$

i.e., $2x - z - 2 = 0$

Intersection of two surfaces could generate a curve of intersection, and we could use the gradients to both surfaces at a point on this curve to find the line tangent to both surfaces at that point.

Say the two surfaces are $f(x,y,z) = c_1$ and $g(x,y,z) = c_2$. The tangent line in question is perpendicular to both $(\nabla f)_{P_0}$ and $(\nabla g)_{P_0}$.



Prob 15 $\quad \underbrace{x^2+2y+2z=4}_{z = 2-\left(\frac{1}{2}\right)x^2 - y}, \quad y=1, \quad P_0(1,1,\frac{1}{2})$

$f(x,y,z)=c_1 \qquad g(x,y,z)=c_2$

$\nabla f = 2x\,\hat{i} + 2\hat{j} + 2\hat{k} \qquad \nabla g = 0\hat{i} + 1\hat{j} + 0\hat{k} = \hat{j}$

$(\nabla f)_{P_0} = 2\hat{i} + \overset{2\cdot 1}{2}\hat{j} + \overset{2\times 1}{2}\hat{k} \qquad (\nabla g)_{P_0} = \hat{j}$

We need to find a direction orthogonal to both ∇f and ∇g.

$$\bar{v} = \nabla f \times \nabla g = \underbrace{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix}}_{\text{determinant}} = (2\times 0 - 1\times 2)\hat{i} - (2\times 0 - 0\times 2)\hat{j} + (2\times 1 - 0\times 2)\hat{k}$$

$$= -2\hat{i} + 2\hat{k}$$

$P_0(\overset{x_0}{1}, \overset{y_0}{1}, \overset{z_0}{\tfrac{1}{2}})$

The tangent line is $x = x_0 - 2t,\ y = y_0 + 0t,\ z = z_0 + 2t$

i.e., $x = 1-2t,\ y = 1,\ z = \frac{1}{2} + 2t,\quad -\infty < t < \infty.$

We will visualize these two surfaces. While it is not necessary to use a computer to create such visualizations, it could be very instructive.

We could produce visualizations of these two surfaces in Octave/Matlab. It is simpler to create visualizations when the surface is given as $z=f(x,y)$, but certain packages such as Mathematica and Gnuplot could also plot implicitly defined surfaces directly.

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% Problem 15 from Section 13.6
% x^2 + 2y + 2z = 4 and y=1 at P0 (1,1,1/2)

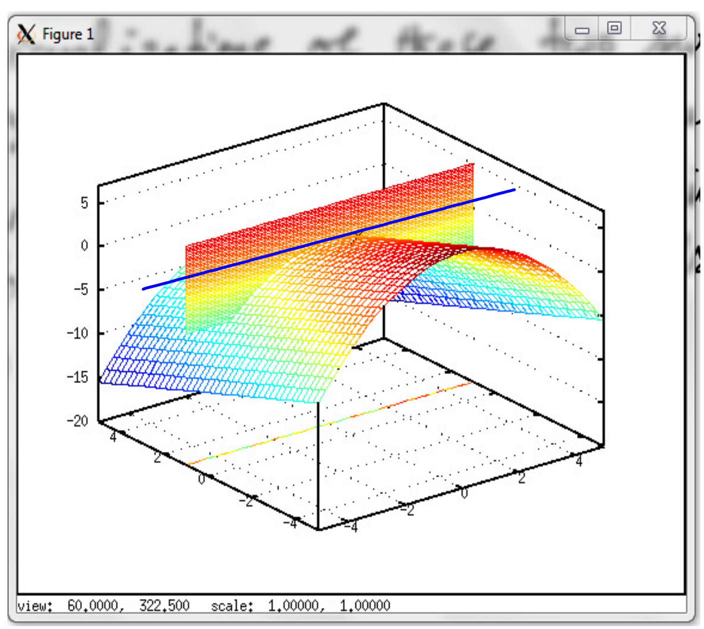
tx = ty = linspace (-5,5,41)';
[xx,yy] = meshgrid (tx,ty);
tz = 2 - (1/2)*xx.^2 - yy;     % z = 2 - (1/2)x^2 - y
mesh(tx,ty,tz);
hold on;

meshc(xx,ones(size(xx)),yy);   % y = 1

plot3(1,1,1/2,"x","markersize",10,"color","k")
```

→ These are the commands in Octave to plot the two surfaces. We generate a regular grid of points (x,y) using meshgrid, and evaluate the function at each such point.

This is one view of the visualization generated. One could rotate the view in Octave. The tangent line is drawn on top here by hand (but one could do it in Octave as well).



Prob 17 $\quad \underbrace{x^3+3x^2y^2+y^3+4xy-z^2}_{f}=0$

$\underbrace{x^2+y^2+z^2}_{g}=11, \quad P_0(1,1,3)$

Find the equation to the line tangent to both surfaces at P_0.

$$\nabla f = (3x^2+6xy^2+4y)\hat{i} + (6x^2y+3y^2+4x)\hat{j} + (-2z)\hat{k}$$

$$\nabla g = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$(\nabla f)_{(1,1,3)} = 13\hat{i}+13\hat{j}-6\hat{k}, \quad (\nabla g)_{(1,1,3)} = 2\hat{i}+2\hat{j}+6\hat{k}$$

$$\bar{v} = \begin{vmatrix} i & j & k \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = 90\hat{i}-90\hat{j}+0\hat{k} = 90\hat{i}-90\hat{j}$$

$$(x_0,y_0,z_0)=(1,1,3)$$

Tangent line : $x = 1+90t$
$y = 1-90t \qquad -\infty < t < \infty$
$z = 3$