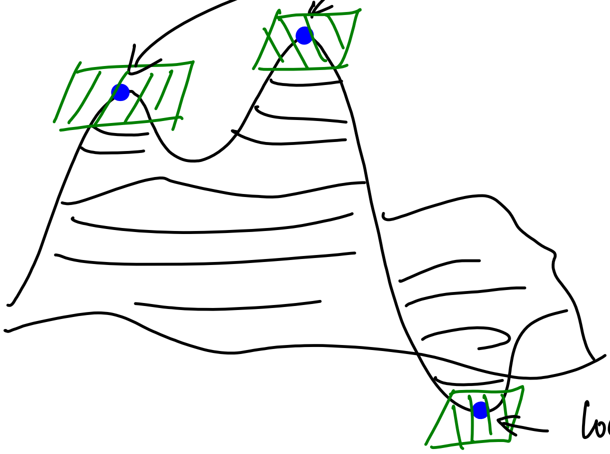
MATH 273 – Lecture 15 (10/14/2014)



Tangent planes at local maxima/minima are horizontal.

Local maxima and local minima are together referred to as local extrema, also called relative extrema.

as opposed to global or absolute extrema.

Indeed, the tangent planes being horizontal at local extrema gives the characterization of these points of interest.

Theorem 10 The first derivative test for local extrema:

If $f(x,y)$ has a local optimum at an interior point (a,b) in its domain, and its first partial derivatives exist at (a,b), then $f_x(a,b)=0$ and $f_y(a,b)=0$.

if f_x or f_y (or both) do not exist, then also (a,b) is a point of interest!

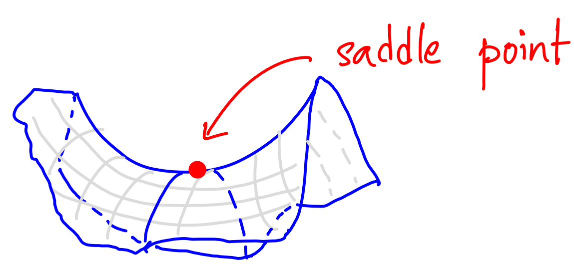
Def An interior point of the domain of $f(x,y)$ where both both f_x and f_y are zero, or where one or both of f_x and f_y do not exist is a **critical point** of $f(x,y)$.

Notice that not all critical points need be local extrema. The partial derivatives could be undefined, or it could be a saddle point, which corresponds to inflection points in 1D.

Def A differentiable function $f(x,y)$ has a **saddle point** at a critical point (a,b) if in every open disk centered at (a,b) there are points (x,y) (in the domain) where $f(x,y) > f(a,b)$, and also other points (u,v) in the domain where $f(u,v) < f(a,b)$.

The corresponding point on the surface $z = f(x,y)$, i.e., $(a,b,f(a,b))$ is a saddle point of the surface.

The name "saddle point" is quite apt.



How do we tell apart the local optima, saddle points, and other critical points?

The Second Derivative Test

Theorem 11 Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b), and $f_x(a,b) = f_y(a,b) = 0$. Then

(i) f has a local maximum at (a,b) if
$f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b);

(ii) f has a local minimum at (a,b) if
$f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b);

(iii) f has a saddle point at (a,b) if
$f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b); and

(iv) the test is inconclusive if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b).

The quantity $f_{xx}f_{yy} - f_{xy}^2$ is called the **Hessian** or **discriminant** of $f(x,y)$.

Hessian as a 2×2 determinant

$$H(f(x,y)) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Recall, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Problem 1 Find local extrema and saddle points, if any, of

$$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$$

$f_x = f_y = 0$ for critical points here (notice, domain is all real pairs).

$$f_x = 2x + y + 0 + 3 - 0 + 0 = 0$$

$$f_y = 0 + x + 2y + 0 - 3 + 0 = 0$$

i.e.,

$$2x + y + 3 = 0 \quad \text{——} \quad (1)$$

$$x + 2y - 3 = 0 \quad \text{——} \quad (2)$$

$2\times(1) - (2)$: $3x + 9 = 0$, i.e, $x = -3$, so $y = 3$

So, $(-3,3)$ is the only critical point.

$$\frac{\partial}{\partial x}(2x+y+3) \qquad \frac{\partial}{\partial y}(x+2y-3) \qquad \frac{\partial}{\partial y}(2x+y+3)$$

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

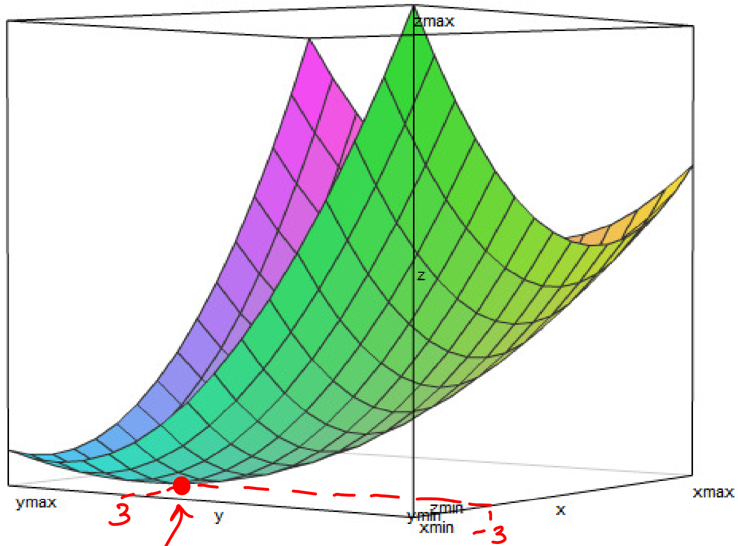
$H = f_{xx}f_{yy} - f_{xy}^2 = 2\times 2 - (1)^2 = 3 > 0$, hence $(-3,3)$ is a local extremum.

Since $f_{xx} = 2 > 0$, $(-3,3)$ is a local minimum of $f(x,y)$.

Also, $f(-3,3) = -5$ is a local minimum of $f(x,y)$.

Let's visualize the surface $z = f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$.

Here, $-6 \le x \le 6$, and $-6 \le y \le 6$



Indeed, $(-3, 3, -5)$ is a local minimum.

Prob 13 $\quad f(x,y) = x^3 - y^3 - 2xy + 6$

Find all local extrema, saddle points, and other critical points of $f(x,y)$.

Domain is all real pairs of values.

$f_x = 3x^2 - 2y = 0$ —— (1)

$f_y = -3y^2 - 2x = 0$ —— (2)

(1) $\Rightarrow$ $y = \frac{3}{2}x^2$. So (2) $\Rightarrow$ $-3\left(\frac{3}{2}x^2\right)^2 - 2x = 0$, i.e.

"implies"

$$x\left(\frac{27}{4}x^3 + 2\right) = 0$$

$x = 0$, or $x^3 = \frac{-8}{27}$, i.e., $x = 0$ or $x = -\frac{2}{3}$, giving

$y = 0$, $y = \frac{3}{2}\left(\frac{-2}{3}\right)^2 = \frac{2}{3}$.

The critical points are $(0,0)$ and $\left(\frac{-2}{3}, \frac{2}{3}\right)$.

$f_{xx} = 6x$, $f_{yy} = -6y$, $f_{xy} = -2$.

$$H = f_{xx}f_{yy} - f_{xy}^2 = (6x)(-6y) - (-2)^2$$
$$= -36xy - 4.$$

$(0,0)$

$H = -36 \cdot 0 \cdot 0 - 4 = -4 < 0$

So $(0,0)$ is a saddle point

$f(0,0) = 6$

$\left(-\frac{2}{3}, \frac{2}{3}\right)$

$H = -36\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) - 4 = 12 > 0$

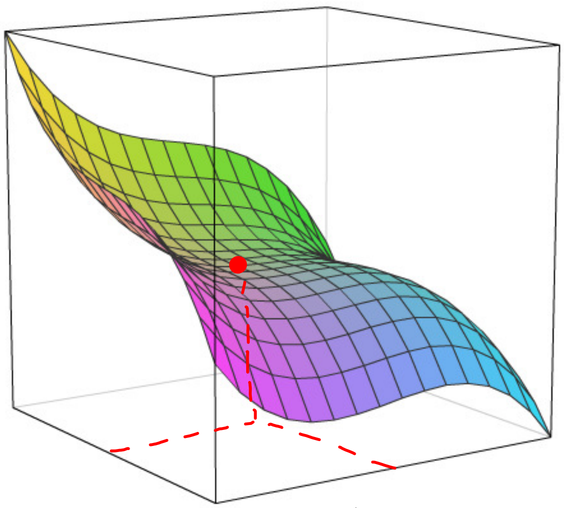
$f_{xx} = 6x = 6\left(\frac{-2}{3}\right) = -4 < 0$

$\left(-\frac{2}{3}, \frac{2}{3}\right)$ is a local maximum.

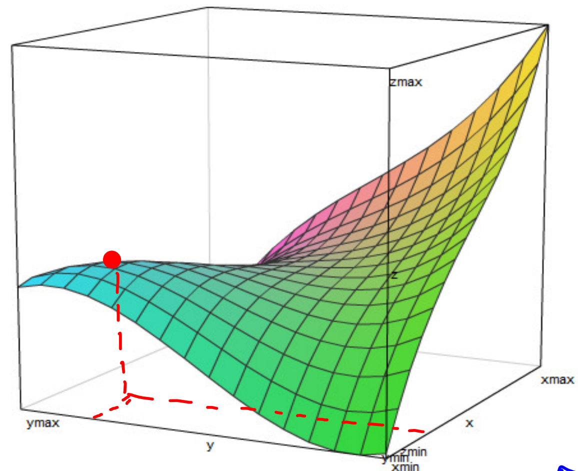
$$f\left(\frac{-2}{3}, \frac{2}{3}\right) = \left(\frac{-2}{3}\right)^3 - \left(\frac{2}{3}\right)^3 - 2\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) + 6$$
$$= \frac{170}{27}.$$

Hence, $(0,0,6)$ is a saddle point on the surface $z = f(x,y) = x^3 - y^3 - 2xy + 6$, while $\left(\frac{-2}{3}, \frac{2}{3}, \frac{170}{27}\right)$ is a local maximum.

Again, let's visualize $z = f(x,y) = x^3 - y^3 - 2xy + 6$.



$(0,0,6)$ – saddle point



$\left(-\frac{2}{3}, \frac{2}{3}, \frac{170}{27}\right)$: local maximum

$-1 \le x \le 1,\ -1 \le y \le 1$