MATH 273 – Lecture 18 (10/23/2014)

Prob 39

Find a, b with $a \le b$ such that

$$\int_a^b (6-x-x^2)\,dx \text{ has its largest value.}$$

Let $F(a,b) = \int_a^b (6-x-x^2)\,dx$ with $a \le b$.

$F(a,b)$ is defined for all pairs of real numbers (a,b) satisfying $a \le b$. The line $a=b$ specifies the boundary of the domain.

Indeed, $F(a,a) = \int_a^a (6-x-x^2)\,dx = 0$, so $F(a,b) = 0$ on all the boundary of its domain. So we look for interior critical points.

$$\frac{\partial F}{\partial a} = -(6-a-a^2) = 0 \quad \text{——— (1)} \qquad (1) \Rightarrow (a+3)(a-2) = 0, \quad a = -3, 2$$

$$\frac{\partial F}{\partial b} = (6-b-b^2) = 0 \quad \text{——— (2)} \qquad \Rightarrow b = -3, 2$$

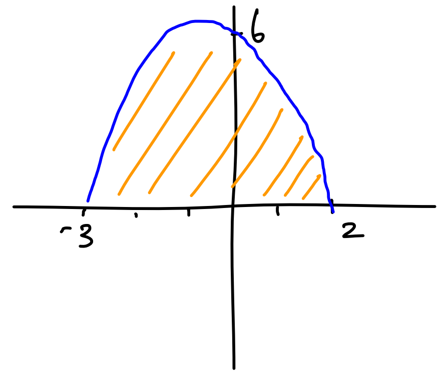
Here, the system of two equations (1) & (2) have independent equations in a and in b, respectively. So, we could consider all possible pairs arising from $a = -3, 2$ and $b = -3, 2$.

But we are looking for interior points satisfying $a \le b$, i.e., $a < b$. So $(-3,2)$ is the only candidate.

$(-3,-3)$ & $(2,2)$ are boundary points

$$F(-3,2) = \int_{-3}^{2} (6 - x - x^2)\,dx$$

This is the area under the parabola $y = 6 - x - x^2$ between $x = -3$ and $x = 2$ above the x-axis.



This area is strictly bigger than zero, while it is zero on the boundary $(a = b)$.

$$F(3,-2) = 6x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_{-3}^{2}$$

you need not compute $F(a,b)$ for the choice of a, b — the problem asks you to just identify the correct pair a, b.

$$= 6(2+3) - \tfrac{1}{2}(4-9) - \tfrac{1}{3}(8+27)$$

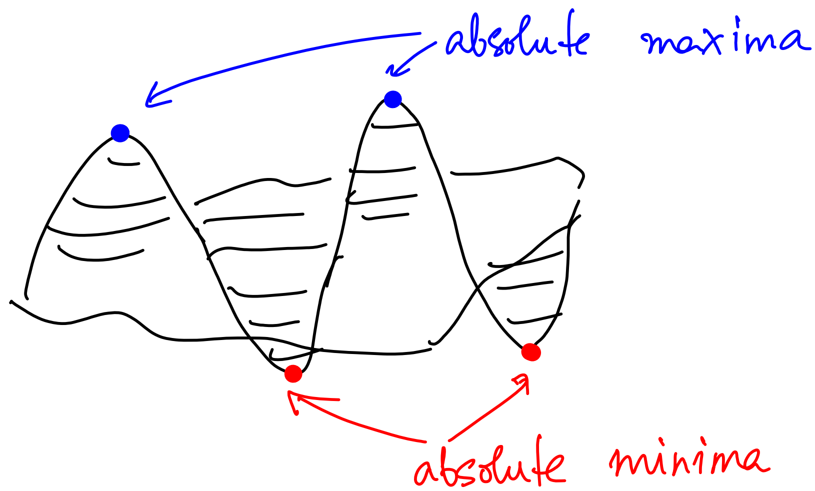
$$= \frac{180 + 15 - 70}{6} = \frac{125}{6}.$$

The other option would be to first compute the integral

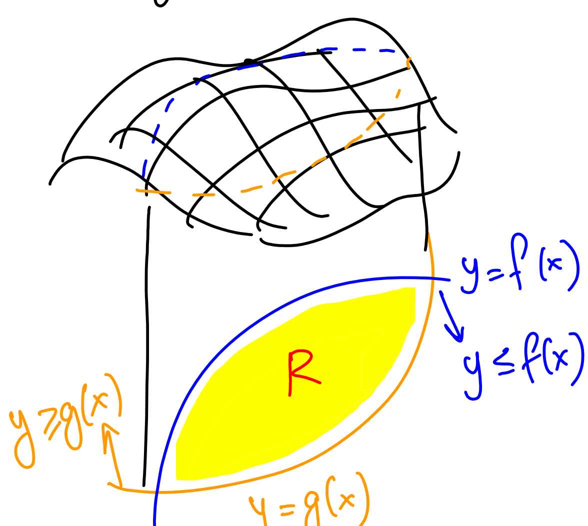
$$F(a,b) = \int_a^b (6 - x - x^2)\,dx = 6(b-a) - \tfrac{1}{2}(b^2 - a^2) - \tfrac{1}{3}(b^3 - a^3),$$

and then find the absolute maximum of $F(a,b)$ using the techniques we employed previously. The result should be identical.

We could have multiple absolute maxima and/or multiple absolute minima!



The question of finding absolute maxima or absolute minima could be posed over very general regions R (and not just triangles or rectangles).



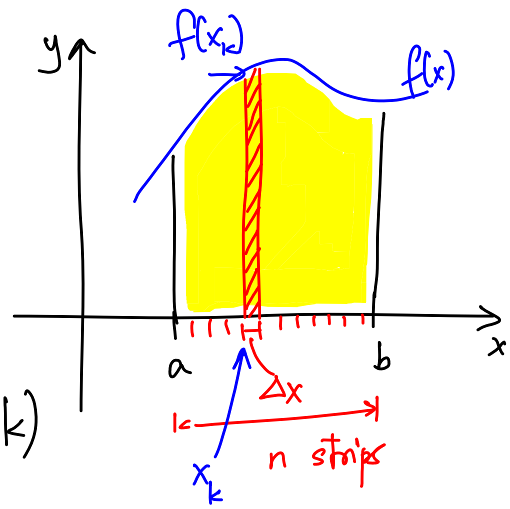
We use the idea of Lagrange multipliers – we'll skip this topic ☺!

Multiple Integrals over Rectangular Domains (Chapter 14)

In 1D

$$\int_a^b f(x)\,dx = \text{Area under curve } y=f(x) \text{ between } x=a \text{ and } x=b \text{ above the } x\text{-axis.}$$

$a \le b$



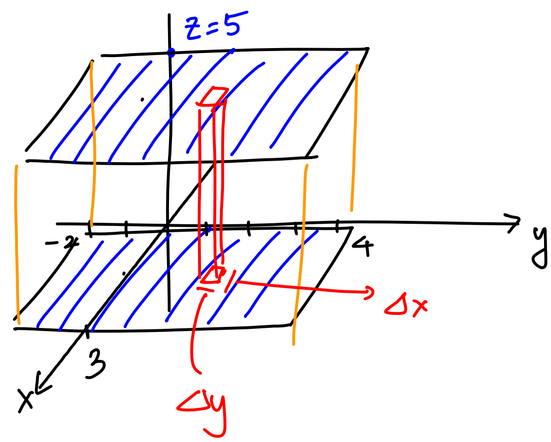
$$= \lim_{n\to\infty} \sum_{k=1}^{n} (\text{area of rectangular strip } k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$= \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

We extend this idea to 2D!

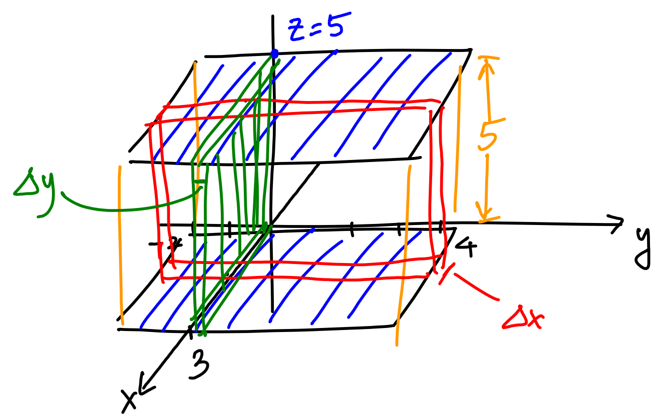
Consider $z=f(x,y)=5$, and the volume under this surface between $0\le x\le 3$, $-2\le y\le 4$, above the xy-plane.



We could evaluate this volume by adding up volumes of thin rectangular columns of widths Δx and Δy.

Volume $V = \lim_{\Delta x \to 0} \sum_{i=1}^{m} 6 \times 5 \times \Delta x$ $\rightarrow f(x,y)$

$= \lim_{\Delta y \to 0} \sum_{j=1}^{n} 3 \times 5 \times \Delta y$



We could have $f(x,y)$ be more general — and not necessarily constant, here!

$$V = \lim_{\Delta x, \Delta y \to 0} \sum f(x,y)\, \Delta x \Delta y = \lim_{\Delta A \to 0} \sum f(x,y) \underbrace{\Delta A}_{\text{small area}}$$

The volume under the surface $z = f(x,y)$ within $a \le x \le b$, $c \le y \le d$, above the xy plane is defined as the **double integral**

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

inner

outer

The order of integration is immaterial as long as $f(x,y)$ is continuous over all of R (Fubini's theorem).