MATH 273 – Lecture 20 (10/30/2014)

Double integration over general domains (14.2)

Theorem 2 (Fubini's stronger theorem)

Let $f(x,y)$ be a continuous function on region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with $g_1(x)$ and $g_2(x)$ are continuous over $x \in [a,b]$, then "element of" or "in"

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with $h_1(y)$ and $h_2(y)$ are continuous over $y \in [c,d]$, then

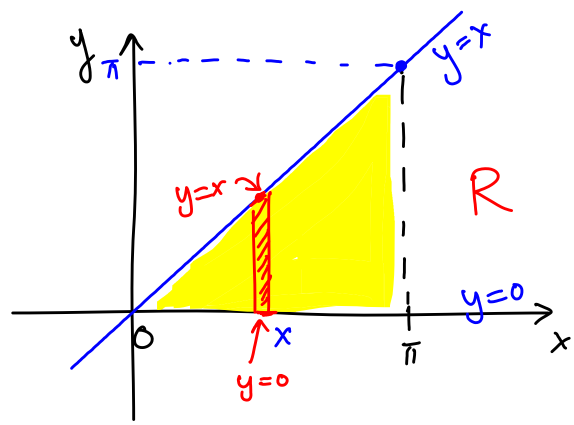
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

For a given integral $\iint_R f(x,y)\,dA$, we could use either of these two forms, and we should get the same answer.

We first try to evaluate such a double integral, and then provide details of how to specify the details of the region of integration.

19. Sketch the region of integration R and evaluate the integral
$$\int_0^{\pi}\int_0^{x} x\sin y\,dy\,dx.$$

The integral is of the form described in Option 1,
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx,$$
with $g_1(x)=0$ and $g_2(x)=x$. Or, the limits of y are $y=0$ to $y=x$, and limits of x are $[0,\pi]$.



More generally, one needs to plot $g_1(x)$ and $g_2(x)$, and decide which sides of these two curves to pick.

$$\iint_R x\sin y\,dA = \int_0^{\pi}\int_0^{x} x\sin y\,dy\,dx = \int_0^{\pi}\left(x\left(-\cos y\right)\Big|_0^x\right)dx$$

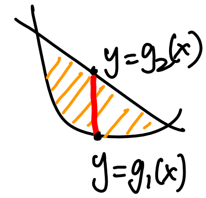
$$= \int_0^{\pi} x\left[-\cos x - -\cos 0\right]dx = \int_0^{\pi} x(1-\cos x)\,dx$$

$$= \int_0^{\pi}(x - x\cos x)\,dx = \left[\tfrac{1}{2}x^2 - (x\sin x + \cos x)\right]\Big|_0^{\pi}$$

$$= \tfrac{1}{2}(\pi^2) - \pi\underbrace{\cancel{\sin\pi}}_{=0} - \underbrace{\cos\pi}_{-1} - \left(\tfrac{1}{2}(0^2) + 0\sin 0 + \underbrace{\cos 0}_{1}\right) = \frac{\pi^2}{2} + 2.$$

Sketching Regions of Integration

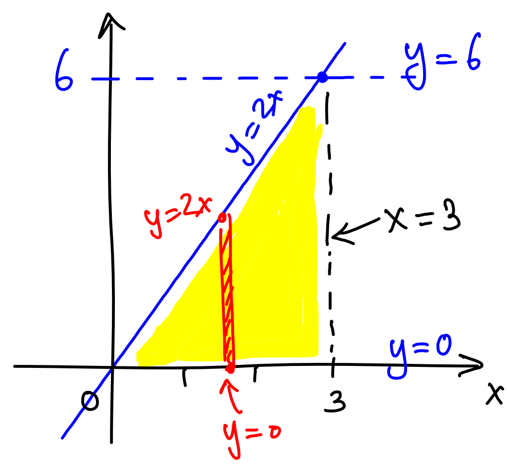
Procedure using vertical cross sections



1. Sketch region and label bounding curves.
2. Imagine a vertical line crossing the region at x, and figure out the limits of y as functions of x.
3. Find the limits for x, such that the region includes all possible vertical lines as used in Step 2.

The procedure using horizontal cross sections is similar, except that the roles of x and y are reversed.

(1). Sketch the region of integration $0 \le x \le 3$, $0 \le y \le 2x$.



Since the limits of y are given as functions of x here, we are indeed using vertical cross sections. But notice that we could equivalently describe the region as

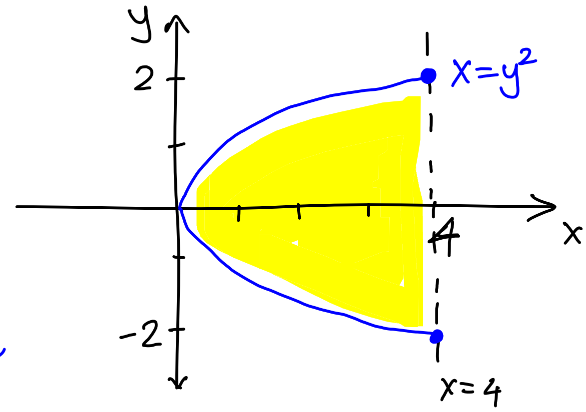
$$0 \le y \le 6, \quad \frac{y}{2} \le x \le 3;$$

using horizontal cross sections.

3. $-2 \le y \le 2$, $y^2 \le x \le 4$

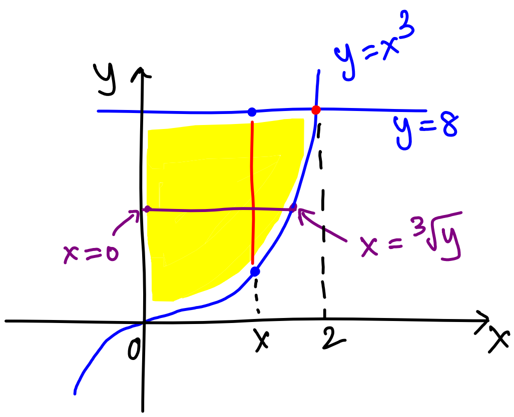
$x = y^2$ gives

$y = \sqrt{x}$



$x = y^2$ has the shape of the parabola $y = x^2$, but with x and y flipped.

(9). Write the integral for $\iint_R dA$ over region R using (a) vertical cross sections and (b) horizontal cross sections.



(a) $\displaystyle\int_0^2 \int_{x^3}^{8} dy\, dx$

Notice that for the vertical line cutting across the region, y varies from x^3 to 8.

Also, $y = x^3$ and $y = 8$ intersect at $(2, 8)$.

(b) $\displaystyle\int_0^8 \int_0^{\sqrt[3]{y}} dx\, dy$

For the horizontal line crossing the region, x varies from 0 to $(y)^{1/3}$, i.e., $\sqrt[3]{y}$.