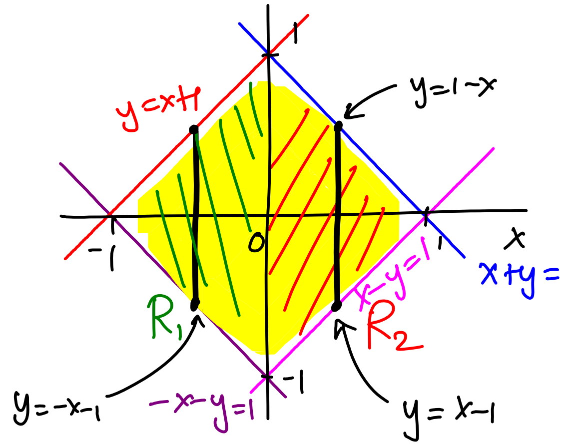
MATH 273 – Lecture 22 (11/06/2014)

55. $$I = \iint_R (y - 2x^2)\,dA$$

$$= \iint_{R_1} (y-2x^2)\,dA + \iint_{R_2} (y-2x^2)\,dA$$



Can use vertical cross sections to write each integral.

$$I = \int_{-1}^{0}\int_{-x-1}^{x+1} (y-2x^2)\,dy\,dx + \int_{0}^{1}\int_{x-1}^{1-x} (y-2x^2)\,dy\,dx$$

$$= \int_{-1}^{0}\left(\frac{1}{2}y^2 - 2x^2 y\,\Big|_{\underbrace{-x-1}_{=-(x+1)}}^{x+1}\right)dx + \int_{0}^{1}\left(\frac{1}{2}y^2 - 2x^2y\,\Big|_{\underbrace{x-1}_{=-(1-x)}}^{1-x}\right)dx$$

$$= \int_{-1}^{0}\left(\frac{1}{2}\left[\cancel{(x+1)^2} - \cancel{(-(x+1))^2}\right] - 2x^2\Big[\underbrace{(x+1) - -(x+1)}_{2(x+1)}\Big]\right)dx$$

$$+ \int_{0}^{1}\left(\frac{1}{2}\left[\cancel{(1-x)^2} - \cancel{(-(1-x))^2}\right] - 2x^2\Big[\underbrace{1-x - -(1-x)}_{2(1-x)}\Big]\right)dx$$

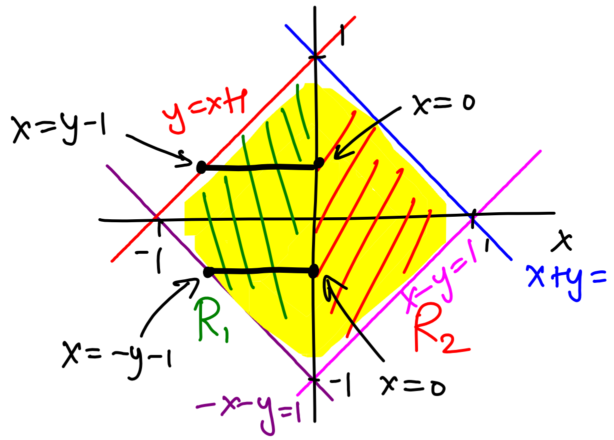
$$= \int_{-1}^{0} -4(x^3+x^2)\,dx + \int_{0}^{1} -4(x^2-x^3)\,dx$$

$$= \left(-x^4 - \frac{4}{3}x^3\right)\Big|_{-1}^{0} + \left(-\frac{4}{3}x^3 + x^4\right)\Big|_{0}^{1} = -\left(-(-1)^4 - \frac{4}{3}(-1)^3\right)$$

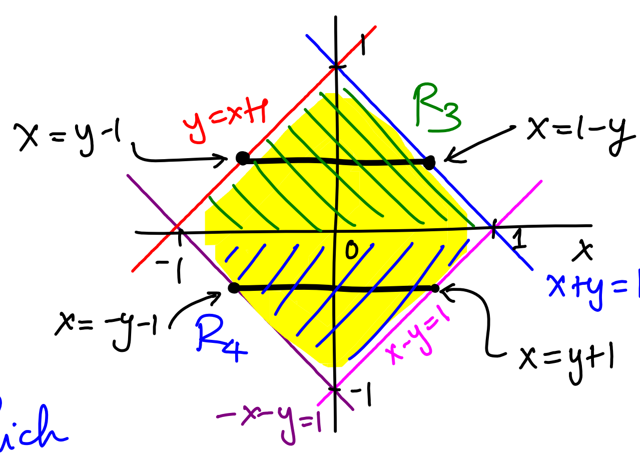
$$+ -\frac{4}{3}(1)^3 + (1)^4$$

$$= 1 - \frac{4}{3} - \frac{4}{3} + 1 = \frac{-2}{3}.$$

We might not want to use horizontal cross sections after splitting R into R_1 and R_2 as we did.



But instead, we could have split R horizontally into R_3 and R_4, say, and then used horizontal cross sections.



Just as we could choose which variable to differentiate first w.r.t. in a second derivative, e.g., $\frac{\partial^2 f}{\partial x \partial y}$, we could choose which variable to integrate w.r.t. in a double integral, so that the computation becomes easier.

Properties of Double Integrals

Let $f(x,y)$ and $g(x,y)$ be continuous functions over region R.

1. $\iint_R c\, f(x,y)\,dA = c\iint_R f(x,y)\,dA$ (constant multiple)

2. Sum/difference

$$\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

3. Domination.

on all points in R

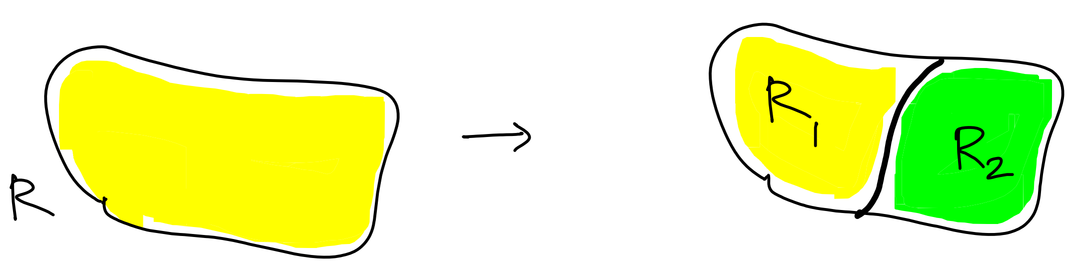
If $f(x,y) \geq g(x,y)$ on R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

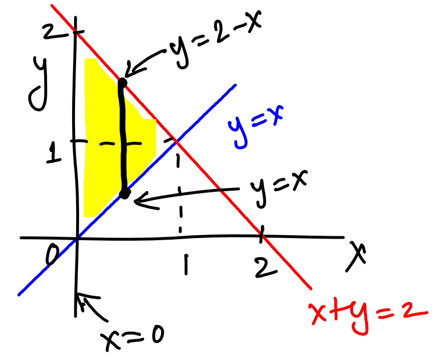
4. Additivity

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$ where

R is the union of nonoverlapping regions R_1 and R_2.



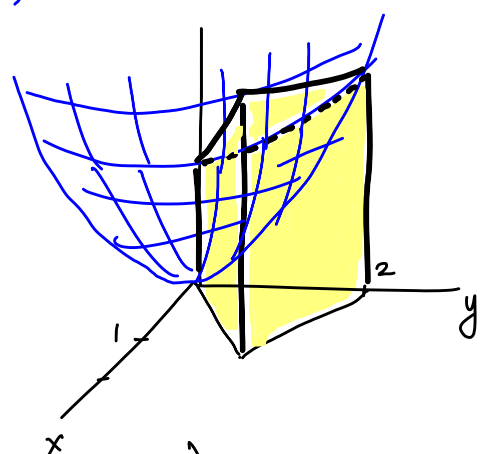
57. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the x-y plane.



$$V = \int_0^1 \int_x^{2-x} (x^2+y^2)\,dy\,dx = \int_0^1 \left(x^2 y + \frac{1}{3}y^3 \Big|_x^{2-x} \right) dx$$

$$= \int_0^1 \Big(x^2 \underbrace{(2-x-x)}_{2-2x} + \frac{1}{3}\big[(2-x)^3 - x^3\big] \Big)\,dx$$

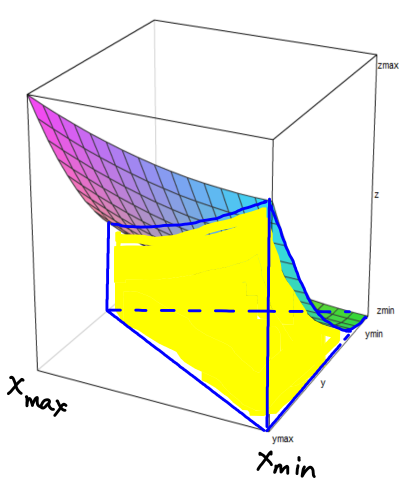
$8 - x^3 - 12x + 6x^2 \longrightarrow (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$



$$= \int_0^1 \Big(\frac{3}{3}(2x^2 - 2x^3) + \frac{1}{3}\big(8 - 12x + 6x^2 - 2x^3\big) \Big)\,dx$$

$$= \frac{1}{3} \int_0^1 \big(8 - 12x + 12x^2 - 8x^3\big)\,dx$$

$$= \frac{1}{3}\Big[8x - 6x^2 + 4x^3 - 2x^4\Big]\Big|_0^1 = \frac{1}{3}(8 - 6 + 4 - 2) = \frac{4}{3}.$$



$0 \le x \le 1$, $0 \le y \le 2$ here, giving $0 \le z \le 5$.

This is a more accurate figure than the one drawn by hand above ☺!