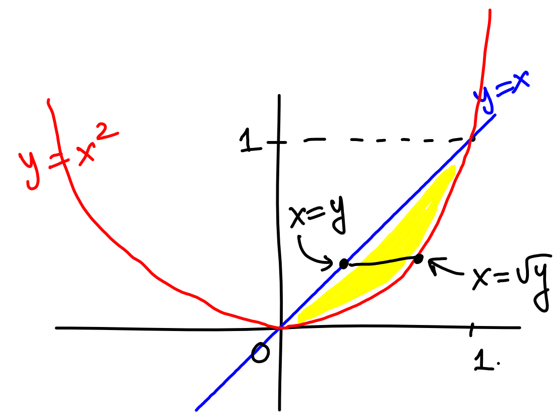
MATH 273 – Lecture 23 (11/11/2014)

Exam 2 Review

7. **(14)** Sketch the region of integration, and write an equivalent integral with the order of integration reversed. Then evaluate this reverse ordered integral.

$$I = \int_0^1 \int_{x^2}^{x} \sqrt{x}\, dy dx.$$



I uses vertical cross sections.

y varies from x^2 to x, and

x varies from 0 to 1.

Points of intersection of $y=x$ and $y=x^2$: $x=x^2$, i.e. $x(x-1)=0$, giving $x=0,1$, for which $y=0,1$.

The points of intersection are $(0,0)$ and $(1,1)$.

Reversing the order of integration, we write

$$I = \int_0^1 \int_y^{\sqrt{y}} \sqrt{x}\, dx\, dy.$$

$$= \int_0^1 \left(\frac{x^{3/2}}{(3/2)} \Big|_y^{\sqrt{y}} \right) dy = \frac{2}{3} \int_0^1 \left((\sqrt{y})^{3/2} - (y)^{3/2} \right) dy$$

$$= \frac{2}{3} \left[\frac{4}{7} y^{7/4} - \frac{2}{5} y^{5/2} \Big|_0^1 \right] = \frac{2}{3} \left[\frac{4}{7}(1) - \frac{2}{5}(1) \right] - 0$$

$$= \frac{2}{3} \left(\frac{4\times 5 - 2\times 7}{7\times 5} \right) = \frac{2 \times \cancel{6}\, 2}{\cancel{3} \times 35} = \frac{4}{35}.$$

23-2

6. **(12)** Evaluate the double integral over the given region R.

$$I = \iint_R xye^{xy^2}\, dA, \qquad R:\ 0 \le x \le 2,\ \ 0 \le y \le 1.$$

$$I = \int_0^2 \int_0^1 xy e^{xy^2}\, dy\, dx$$

$$= \int_0^2 \left(\frac{1}{2} e^{xy^2} \Big|_0^1 \right) dx$$

$$= \frac{1}{2} \int_0^2 \left[e^{x(1)^2} - e^{x(0)} \right] dx = \frac{1}{2} \int_0^2 (e^x - 1)\, dx$$

$$= \frac{1}{2} (e^x - x) \Big|_0^2 = \frac{1}{2} \left[e^2 - 2 - (\cancel{e^0}^{\,1} - 0) \right] = \frac{1}{2}(e^2 - 3).$$

Notice that

$$\frac{\partial}{\partial y}\left(e^{xy^2}\right) = e^{xy^2} \frac{\partial}{\partial y}(xy^2)$$

$$= x(2y) \cdot e^{xy^2}$$

$$= 2xy e^{xy^2}$$

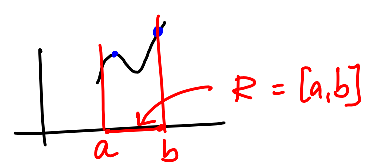
Notice $\frac{\partial}{\partial x}\left(e^{xy^2}\right) = e^{xy^2} \cdot y^2 = y^2 e^{xy^2} \longrightarrow$ so, integrating first w.r.t. x is much harder here!

8. **(6)** Decide whether each of the following statements is *True* or *False*. **Justify** your answer.

(a) A point that gives the absolute maximum of a function in a given region R must also be a local maximum of the function.

(b) Swapping the lower and upper limits of both integrals in a double integral leaves the value of the double integral unchanged.

(a) FALSE. The absolute maximum could occur on the boundary of R.



(b) TRUE. Each swap multiplies the integral by -1, so the value is unchanged as $(-1) \times (-1) = 1$.

3. **(12)** Let $y = uv$. If u is measured with an error of 2% and v is measured with an error of 3%, estimate the percentage error in the calculated value of y.

$y = uv$

The total differential of y is

$$\frac{1}{y}\left(dy = u\,dv + v\,du\right).$$

We want $\frac{dy}{y}$, given $\frac{du}{u} = 2\%$, $\frac{dv}{v} = 3\%$

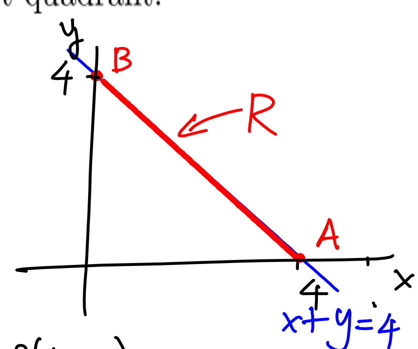
$$\frac{dy}{y} = \frac{u\,dv}{y} + \frac{v\,du}{y}$$

Equivalently, $\frac{du}{u} \times 100 = 2$, $\frac{dv}{v} \times 100 = 3$.

$y = uv$ gives

$$\frac{dy}{y} = \frac{\cancel{u}\,dv}{\cancel{u}v} + \frac{\cancel{v}\,du}{u\cancel{v}} = \frac{dv}{v} + \frac{du}{u} = 3\% + 2\% = 5\%.$$

5. **(16)** Find the absolute maximum and minimum values of $f(x, y) = x^2 + xy + y^2 - 3x + 3y$ on the region R that is the part of the line $x + y = 4$ lying in the first quadrant.



R cannot have interior critical points.

R is $\overline{AB}$ from $A(4,0)$ to $B(0,4)$.

On $\overline{AB}$, $y = 4 - x$, hence

$$f(x, 4-x) = f(x) = x^2 + x(4-x) + (4-x)^2 - 3x + 3(4-x)$$

$$= \cancel{x^2} + 4x - \cancel{x^2} + x^2 - 8x + 16 - 3x + 12 - 3x$$

$$= x^2 - 10x + 28$$

$f'(x) = 2x - 10 = 0$ gives $x = 5$, giving $y = 4 - 5 = -1$.

But $(5,-1)$ is not on $\overline{AB}$. So we just check $f(x,y)$ at $A(4,0)$ and $B(0,4)$.

$f(x,y) = x^2 + xy + y^2 - 3x + 3y$

A: $f(4,0) = (4)^2 + 0 + 0 - 3(4) + 0 = 4 \leftarrow$ absolute minimum

B: $f(0,4) = (0)^2 + 0 + (4)^2 - 0 + 3\times(4) = 28 \leftarrow$ absolute maximum.

4. **(14)** Find all local minima, local maxima, and saddle points of the function given below. You should evaluate the function at each critical point.

$$f(x, y) = x^3 + y^3 - 3xy + 15.$$

The domain is all of $\mathbb{R}^2$ (all real pairs).

Critical points $f_x = 3x^2 - 3y = 0$ ——— (1)

$f_y = 3y^2 - 3x = 0$ ——— (2)

(2) $\Rightarrow$ $x = y^2$. Plugging into (1) gives

$3(y^2)^2 - 3y = 0 \Rightarrow y^4 - y = 0 \quad y(y^3 - 1) = 0$,

giving $y = 0, 1$, and hence $x = 0, 1$, So the critical points are $(0,0)$ and $(1,1)$.

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$; So

$H = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9.$

$(0,0)$

$H = 36(0)(0) - 9 = -9 < 0$

$\Rightarrow (0,0)$ is a saddle point.

$f(0,0) = 15.$

$(0,0,15)$ is a saddle point.

$(1,1)$

$H = 36(1)(1) - 9 = 27 > 0.$

$f_{xx} = 6(1) = 6 > 0. \Rightarrow (1,1)$ is a local minimum.

$f(1,1) = (1)^3 + (1)^3 - 3(1)(1) + 15 = 14.$

$(1,1,14)$ is a local minimum.