MATH 273 – Lecture 24 (11/18/2014)

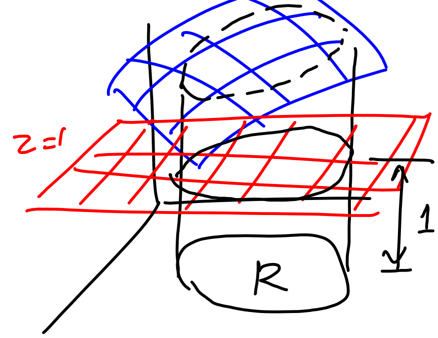
Same offer for final as I made for Exam 2

– If you do really well in the Final, its score can replace (to a large extent) the lower scores of Exams 1 and 2.

Area by Double Integration

We saw $\iint_R f(x,y)\,dA$ gives the volume bounded by $z = f(x,y)$ surface above and R on the xy plane below.

What if $z = f(x,y) = 1$?



Volume $V = \iint_R 1\,dA = \text{Area} \times 1$

Volume of the 3D solid when $f(x,y) = 1$ is just the area × height

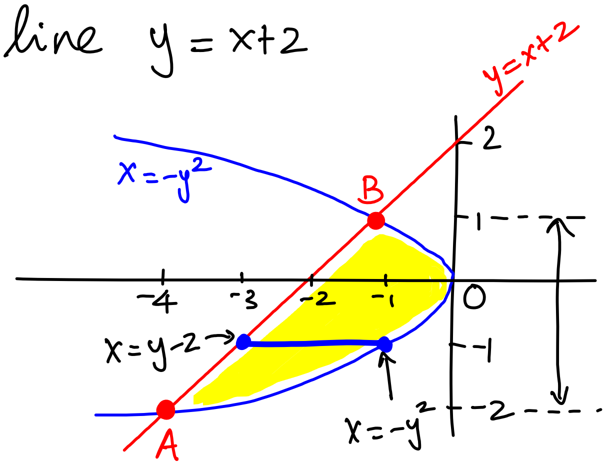
Hence $\text{Area} = \iint_R dA.$

The area of a closed bounded region R in the plane is

$$A = \iint_R dA.$$

③ Sketch the region R bounded by given lines and curves, express its area as a double integral, and evaluate it to find the area.

Parabola $x=-y^2$, line $y=x+2$



y varies from -2 to 1

Points of intersection:

Plug $x=-y^2$ into $y=x+2$

$y=-y^2+2$

$y^2+y-2=0$

$(y+2)(y-1)=0 \Rightarrow y=-2,\ y=1$

$\Rightarrow x=-4,\ -1$

$A(-4,-2)$ and $B(-1,1)$

$$A=\int_{-2}^{1}\int_{y-2}^{-y^2} dx\,dy = \int_{-2}^{1}\left(x\Big|_{y-2}^{-y^2}\right)dy = \int_{-2}^{1}\left[-y^2-(y-2)\right]dy$$

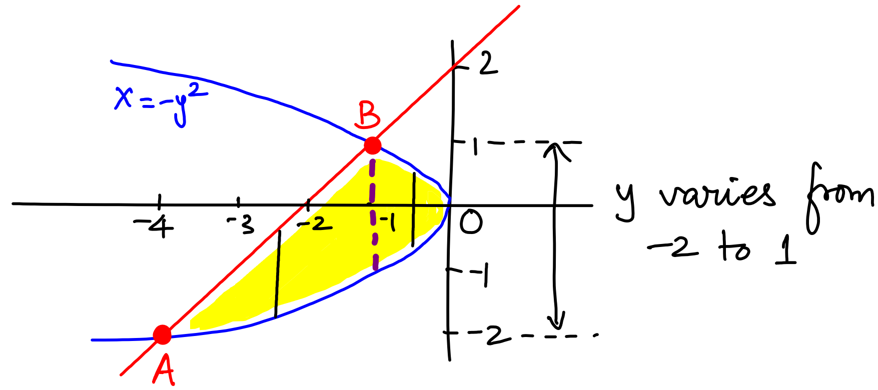
$$=\int_{-2}^{1}(2-y-y^2)\,dy = 2y-\frac{1}{2}y^2-\frac{1}{3}y^3\Big|_{-2}^{1}$$

$$=2\underbrace{(1--2)}_{3}-\frac{1}{2}\underbrace{\left(1^2-(-2)^2\right)}_{-3}-\frac{1}{3}\underbrace{\left(1^3-(-2)^3\right)}_{9}$$

$$=2\times3+\frac{3}{2}-\frac{9}{3}=$$

$$=6+\frac{3}{2}-3=\frac{9}{2}.$$

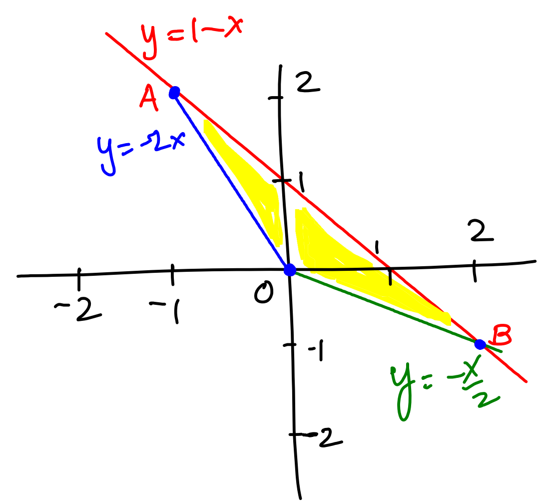
Notice that using vertical cross sections to evaluate the integral would require a split of the region into two — one with $-4 \le x \le -1$, and the other with $-1 \le x \le 0$.



Using horizontal cross sections, we can evaluate the integral in one step.

17. Sketch region of integration, then find area.

$$A = \int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-x/2}^{1-x} dy\,dx$$



1. $y = -2x$ to $y = 1-x$
2. $y = -x/2$ to $y = 1-x$

Points of intersection:

$-2x = 1-x \implies x=-1,\ y=2.\quad A(-1,2)$

$-\frac{x}{2} = 1-x \implies \frac{x}{2} = 1$, i.e., $x=2,\ y=-1$.
$B(2,-1)$.

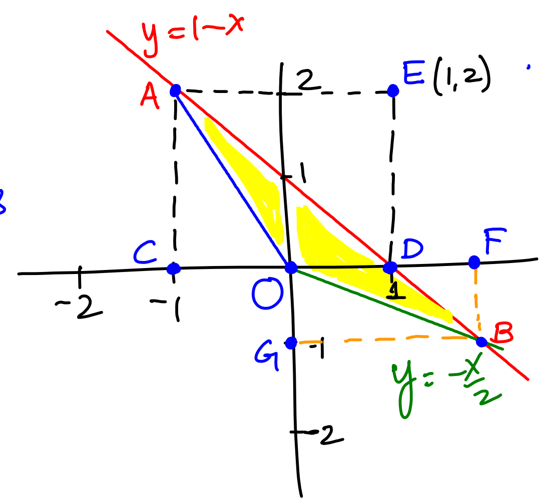
$$A = \int_{-1}^{0} y \Big|_{-2x}^{1-x} dx + \int_{0}^{2} y \Big|_{-x/2}^{1-x} dx$$

$$= \int_{-1}^{0} (1-x - -2x)\,dx + \int_{0}^{2} \left(1-x - -\tfrac{x}{2}\right) dx = \int_{-1}^{0} (1+x)\,dx + \int_{0}^{2} \left(1-\tfrac{x}{2}\right) dx$$

$$= x + \frac{x^2}{2}\Big|_{-1}^{0} + x - \frac{x^2}{4}\Big|_{0}^{2} = (0 - -1) + \tfrac{1}{2}\left(0 - (-1)^2\right) + (2-0) - \tfrac{1}{4}\left((2)^2 - (0)^2\right)$$

$$= 1 - \frac{1}{2} + 2 - 1 = \frac{3}{2}.$$

In this case, the region is simple enough for us to compute the area directly using geometric calculations – just to verify the result from integration. The total area is the sum of the areas of $\triangle OAD$ and $\triangle OBD$



Area of $\triangle OAD$ = Area of $\square ACDE$ − Area of $\triangle OAC$ − Area of $\triangle ADE$

$$= 2 \times 2 - \tfrac{1}{2}(1)(2) - \tfrac{1}{2}(2)(2) = 4 - 1 - 2 = 1.$$

Area of $\triangle OBD$ = Area of $\square OGBF$ − Area of $\triangle OBG$ − Area of $\triangle BDF$

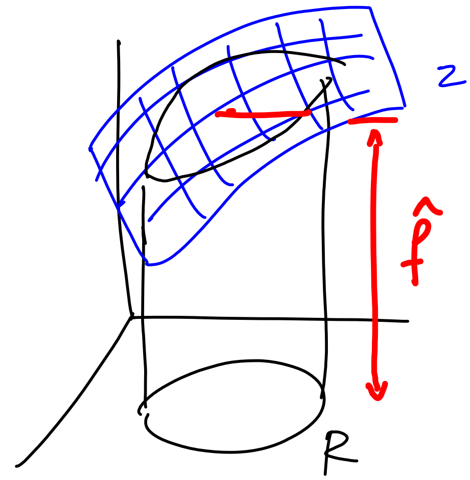
$$= 1 \times 2 - \tfrac{1}{2}(1)(2) - \tfrac{1}{2}(1)(1) = 2 - 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Hence the total area of the region $= 1 + \frac{1}{2} = \frac{3}{2}$.

Average Value of f over R

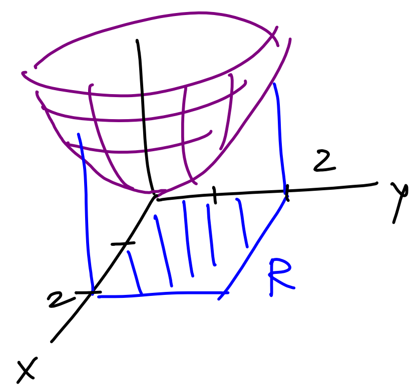
$$\iint_R f\,dA = \text{Volume} = \text{Area}(R) \times \hat{f}$$

$$\Rightarrow \quad \hat{f} = \frac{1}{\text{Area}(R)} \iint_R f\,dA$$



21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

Area $= 2 \times 2 = 4$.



$$\hat{f} = \frac{1}{\text{Area}} \iint_R f(x,y)\,dA$$

$$= \frac{1}{4} \int_0^2 \int_0^2 (x^2+y^2)\,dy\,dx$$

$$= \frac{1}{4} \int_0^2 \left(x^2 y + \frac{1}{3} y^3 \Big|_0^2 \right) dx = \frac{1}{4} \int_0^2 \left(2x^2 + \frac{8}{3} \right) dx$$

$$= \frac{1}{4} \left(\frac{2}{3} x^3 + \frac{8}{3} x \right) \Big|_0^2 = \frac{1}{4} \left(\frac{2}{3}(2)^3 + \frac{8}{3}(2) \right) = \frac{8}{3}.$$