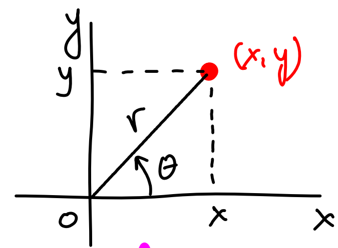
MATH 273 – Lecture 25 (11/20/2014)

Double Integrals in Polar Form (Section 14.4)

Recall: polar coordinates – r, θ

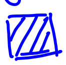
$$(x,y) \equiv (r\cos\theta,\ r\sin\theta)$$

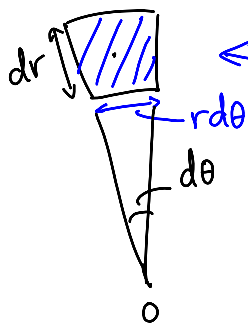
$$r = \sqrt{x^2+y^2}$$

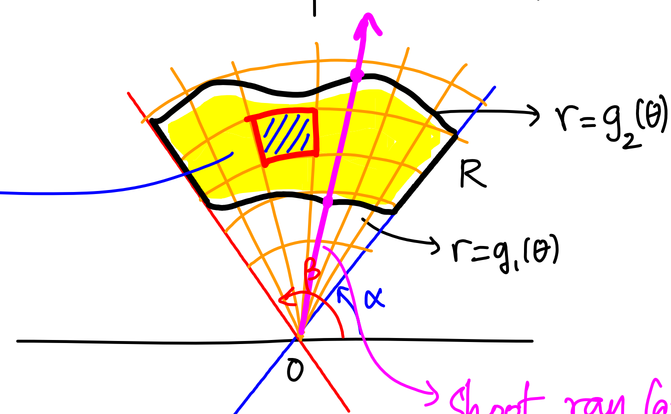


$$\iint_R f(x,y)\, dA$$

$dA \to$ $dx\,dy$ or $dy\,dx$







Shoot ray (arrow) out from origin through R - $g_1(\theta)$ and $g_2(\theta)$ are found at points of entry and exit from R.

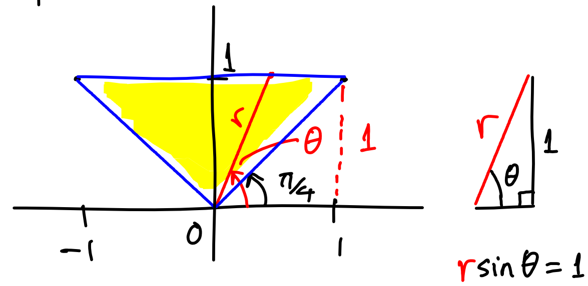
In polar coordinates, $dA = (r\,d\theta)\cdot dr = r\,dr\,d\theta$ → NOT $dr\,d\theta$!

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta.$$

3. Describe region R in polar coordinates.

$$\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$

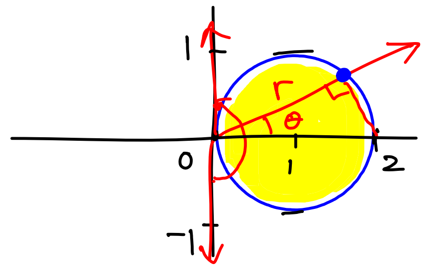
$$0 \le r \le \operatorname{cosec}\theta$$



Describe in polar coordinates

7. The region enclosed by the circle $x^2+y^2=2x$.

$$x^2+y^2=2x$$
$$x^2-2x+1+y^2=1$$
$$(x-1)^2+y^2=1$$



plug in $x=r\cos\theta$, $y=r\sin\theta$ to get

$r^2=2r\cos\theta \implies r=2\cos\theta$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 2\cos\theta.$$

Notice that this computation is equivalent to the ray-shooting method. In the previous example, we can do a similar computation on $y=1$.

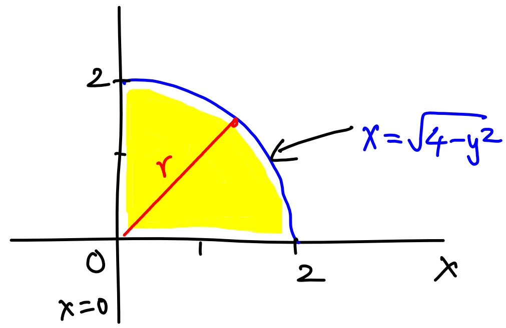
Finding limits of integration in polar coordinates

1. Sketch region.
2. Find r limits (shoot ray (arrow) from origin – find $r=g_1(\theta)$ where it enters R, and $r=g_2(\theta)$ where it leaves R).
3. Find θ limits.

11. Change the Cartesian integral to equivalent polar integral. Then evaluate the polar integral.

$$I = \int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$$

uses horizontal cross sections.



x : 0 to $\sqrt{4-y^2}$

$x = \sqrt{4-y^2} \iff x^2+y^2=4$

$0 \le \theta \le \pi/2$, $0 \le r \le 2$

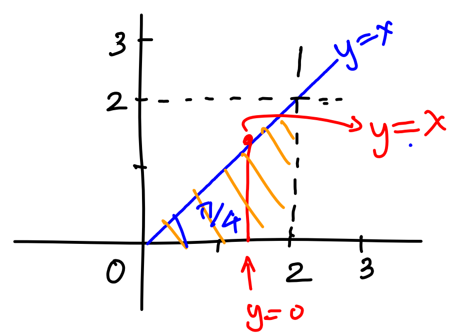
In polar form

$$I = \int_0^{\pi/2}\int_0^2 \underbrace{r^2}_{x^2+y^2} r\,dr\,d\theta = \int_0^{\pi/2}\left(\int_0^2 r^3\,dr\right)d\theta = \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^2 d\theta$$

$$= \frac{16}{4}\int_0^{\pi/2} d\theta = 4\cdot\theta\Big|_0^{\pi/2} = 4\cdot\frac{\pi}{2} = 2\pi.$$

25. Sketch region R, and convert integral to Cartesian form.

$$I = \int_0^{\pi/4}\int_0^{2\sec\theta} r^5\sin^2\theta\,dr\,d\theta$$



At $\theta=0$, $2\sec\theta = 2$

$\theta = \pi/4$ $2\sec\theta = \dfrac{2}{1/\sqrt{2}} = 2\sqrt{2}$

$0 \le x \le 2$

$0 \le y \le x$

$$I = \iint_R r^4 \sin^2\theta \underbrace{r\,dr\,d\theta}_{dy\,dx} = \iint_R \underbrace{r^2}_{(x^2+y^2)} \underbrace{r^2\sin^2\theta}_{y^2}\, r\,dr\,d\theta$$

$$= \iint_R (x^2+y^2)\, y^2\, dy\,dx = \int_0^2\int_0^x (x^2+y^2)\, y^2\, dy\, dx.$$

Area in Polar Coordinates

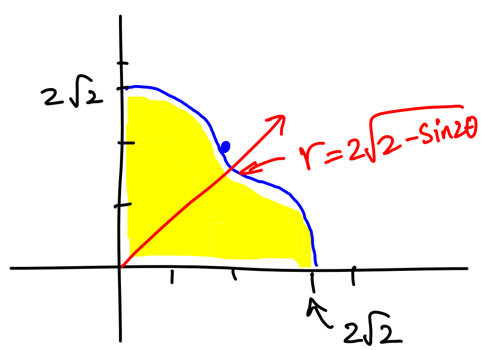
$$A = \iint_R 1 \cdot r\,dr\,d\theta = \iint_R r\,dr\,d\theta$$

29. Find area of region cut from the first quadrant by the curve $r = 2(2-\sin 2\theta)^{1/2} = 2\sqrt{2-\sin 2\theta}$.

$0 \le \theta \le \pi/2$

$r = 2\sqrt{2-\sin 2\theta}$

θ	r
0	$2\sqrt{2}$
$\pi/2$	$2\sqrt{2}$
$\pi/4$	2



$0 \le \theta \le \pi/2$

$0 \le r \le 2\sqrt{2-\sin 2\theta}$

$$A = \int_0^{\pi/2} \int_0^{2\sqrt{2-\sin 2\theta}} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} r^2 \Big|_0^{2\sqrt{2-\sin 2\theta}} \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 4(2-\sin 2\theta)\,d\theta \;=\; 4\theta \Big|_0^{\pi/2} + \cos 2\theta \Big|_0^{\pi/2}$$

$$= 4\left(\frac{\pi}{2} - 0\right) + \left(\cos\pi - \cos 0\right) = 2\pi + (-1-1)$$

$$= 2\pi - 2 = 2(\pi - 1).$$

We will not talk about triple integrals or polar coordinates in 3D due to time constraints. These topics extend the ideas we introduced in 2D to 3D.