MATH 273 – Lecture 26 (12/02/2014)

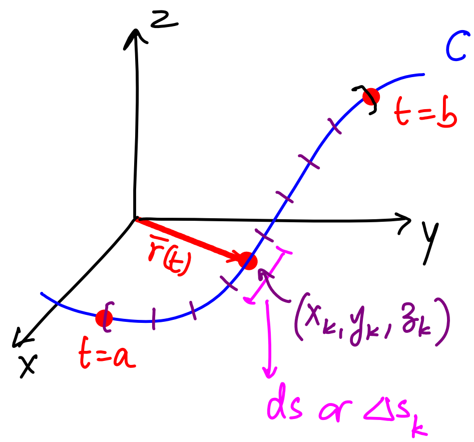
Integration in Vector Fields (Chapter 15)

Rather than integrating over a region in $\mathbb{R}^2$ (2D space) or in $\mathbb{R}^3$ (3D space), we now integrate over a curve or over a surface.

We could use this concept to, for instance, calculate the work done in moving an object along a curved road (curve in 3D), or to find the mass of a curved metallic spring whose density is varying.

Today, we study line integrals.

Line Integrals



Def Let $f(x,y,z)$ be defined on a curve C given by

$$\bar{r}(t) = \underbrace{g(t)}_{x}\,\hat{\imath} + \underbrace{h(t)}_{y}\,\hat{\jmath} + \underbrace{\ell(t)}_{z}\,\hat{k}, \quad a \le t \le b.$$

Then the line integral of f over C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k$$

→ same idea as used in single and double integrals – break region of integration R into small chunks, and evaluate the sum $\sum f(x_k, y_k, z_k)\,dA_k$, for instance.

How do we find ds?

With $\bar{v}(t) = \dfrac{d\bar{r}}{dt}$, we can write $\mathbf{ds = |\bar{v}(t)|\,dt}$, since $\dfrac{ds}{dt} = |\bar{v}(t)|$

"velocity vector"

To evaluate $\displaystyle\int_C f(x,y,z)\,ds$, we

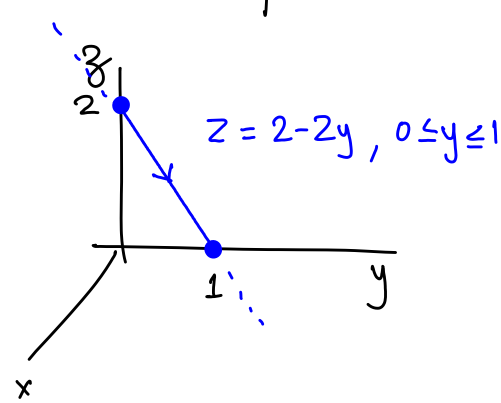
1. find smooth parametrization of C in the form
$\bar{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j} + \ell(t)\,\hat{k}$, $a \le t \le b$, and

2. evaluate
$$\int_C f(x,y,z)\,ds = \int_a^b f\big(g(t), h(t), \ell(t)\big)\,|\bar{v}(t)|\,dt.$$

assume $|\bar{v}(t)| > 0$ over $a \le t \le b$

Probs 1-8

(b) Find parametric expression of the curve in picture.



$y = t$, $0 \le t \le 1$ and
$z = 2-2t$. So, the curve is

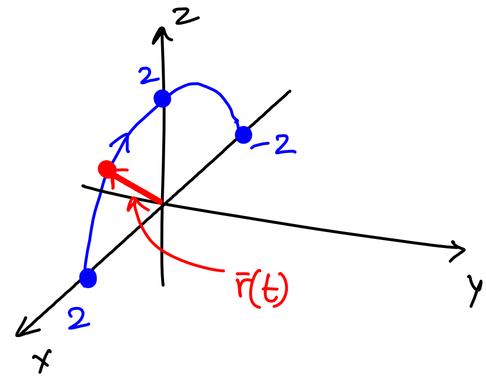
$\bar{r}(t) = t\,\hat{j} + (2-2t)\,\hat{k}$, $0 \le t \le 1$.

given as Problem (6).

(h) $x = 2\cos t$, $z = 2\sin t$, $0 \le t \le \pi$

The parametric curve is

$$\bar{r}(t) = (2\cos t)\,\hat{i} + (2\sin t)\,\hat{k}, \quad 0 \le t \le \pi$$



9. Evaluate $\int_C (x+y)\,ds$ where C is the straight line segment $x = t$, $y = 1-t$, $z = 0$ from $(0,1,0)$ to $(1,0,0)$.

$a \le t \le b$

$$\bar{r}(t) = t\,\hat{i} + (1-t)\,\hat{j}, \quad 0 \le t \le 1.$$

$$f(x,y,z) = x+y = t + (1-t) = 1$$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = 1\cdot\hat{i} + (-1)\,\hat{j} = \hat{i} - \hat{j}. \quad \text{So } |\bar{v}(t)| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

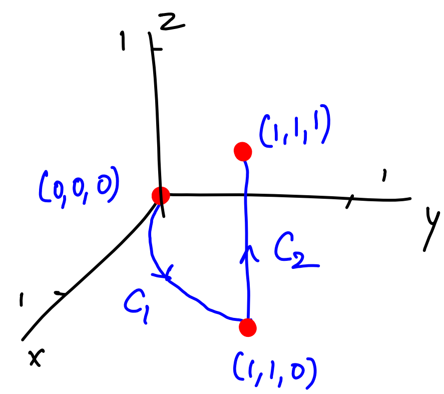
$$\text{So } \int_C f\,ds = \int_0^1 \underbrace{1}_{f} \cdot \underbrace{\sqrt{2}}_{|\bar{v}(t)|}\,dt = \sqrt{2}\,t\Big|_0^1 = \sqrt{2}.$$

In this problem, the parametric form is given to you. In some other problems, you have to find the parametric expression, and then evaluate the integral.

15. Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over path from $(0,0,0)$ to $(1,1,1)$ given by

$$C_1:\ \bar{r}(t) = \underset{x}{t}\,\hat{i} + \underset{y}{t^2}\,\hat{j}, \quad 0 \le t \le 1,$$

$$C_2:\ \bar{r}(t) = \underset{x}{\hat{i}} + \underset{y}{\hat{j}} + \underset{z}{t\hat{k}}, \quad 0 \le t \le 1.$$



$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds,$$ when C is the nonoverlapping union of C_1 and C_2.

$$C_1:\ \bar{v}(t) = 1\hat{i} + 2t\hat{j} \qquad |\bar{v}(t)| = \sqrt{1+4t^2}$$

$$\int_{C_1} f\,ds = \int_0^1 \underset{x\,+\sqrt{y}\,-\,z^2}{\left(t + \sqrt{t^2} - 0^2\right)} \sqrt{1+4t^2}\,dt = \int_0^1 2t\sqrt{1+4t^2}\,dt$$

$$= \frac{1}{4}\cdot\frac{2}{3}\left(1+4t^2\right)^{3/2}\Big|_0^1 = \frac{1}{6}\left(5\sqrt{5}-1\right).$$

$$C_2:\ \bar{v}(t) = \frac{d\bar{r}}{dt} = 0\hat{i} + 0\hat{j} + 1\hat{k} = \hat{k}, \quad \text{So } |\bar{v}(t)| = 1.$$

$$\int_{C_2} f\,ds = \int_0^1 \underset{x+\sqrt{y}-z^2}{\left(1 + \sqrt{1} - t^2\right)}\, 1.dt = \int_0^1 \left(2 - t^2\right) dt = 2t - \frac{1}{3}t^3\Big|_0^1 = \frac{5}{3}.$$

$$\text{So } \int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \frac{1}{6}\left(5\sqrt{5}-1\right) + \frac{5}{3} = \frac{5\sqrt{5}}{6} + \frac{3}{2}.$$