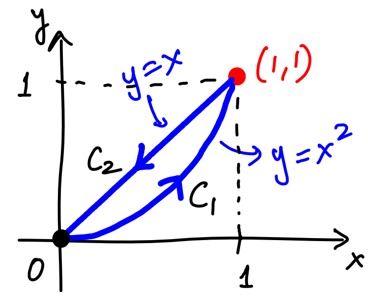
MATH 273 – Lecture 27 (12/04/2014)

Line integrals, continued... (Section 15.1)

(25) Evaluate $\int_C (x+\sqrt{y})\,ds$ where C is given in the picture.



$C_1: y = x^2$ for $0 \le x \le 1$

$$\bar{r}(t) = \underset{x}{\underline{t}}\,\hat{i} + \underset{y}{\underline{t^2}}\,\hat{j}, \quad 0 \le t \le 1$$

$C_2: y = x$ for $1 \le x \le 0$

$$\bar{r}(t) = \underbrace{(1-t)}_{x}\hat{i} + \underbrace{(1-t)}_{y}\hat{j}, \quad 0 \le t \le 1$$

$$C_1: \quad \bar{v}(t) = 1\cdot\hat{i} + 2t\,\hat{j} \implies |\bar{v}(t)| = \sqrt{1^2+(2t)^2} = \sqrt{1+4t^2}$$

$$f(x(t), y(t)) = x + \sqrt{y} = t + \sqrt{t^2} = 2t$$

$$\int_C f(x,y)\,ds = \int_0^1 f(t)\,|\bar{v}(t)|\,dt = \int_0^1 2t\sqrt{1+4t^2}\,dt = \frac{1}{4}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^1$$

$$= \frac{1}{6}\left(5\sqrt{5}-1\right).$$

$$C_2: \quad \bar{v}(t) = -\hat{i} - \hat{j} \implies |\bar{v}(t)| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$$

$$\int_{C_2} f\,ds = \int_0^1 \Big(\underbrace{1-t}_{x} + \underbrace{\sqrt{1-t}}_{\sqrt{y}}\Big)\sqrt{2}\,dt = \int_1^0 (u+\sqrt{u})\sqrt{2}\,(-du)$$

swap limits to cancel −ve sign

$u = 1-t$
$du = -dt$

$$= \int_0^1 (u+\sqrt{u})\sqrt{2}\,du$$

$$= \sqrt{2}\left(\frac{1}{2}u^2 + \frac{2}{3}u^{3/2}\right)\Big|_0^1 = \sqrt{2}\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{7\sqrt{2}}{6}.$$

Hence $\displaystyle\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \frac{(5\sqrt{5}-1)}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5}+7\sqrt{2}-1}{6}.$

Line Integrals over Vector Fields

We now extend the idea of line integrals to the settings of a vector field. A gravitational field, or electromagnetic field are typical examples — we will study how to compute the work done in moving an object along a curve in such a field.

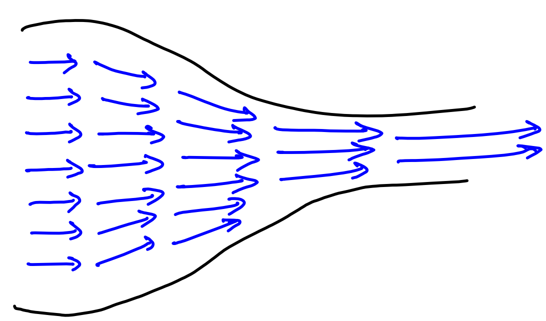
Def A **vector field** is a function that assigns a vector to each point in its domain.

$$F(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

F is continuous (differentiable) if $M, N,$ and P are continuous (differentiable).

Examples

1. fluid flow through a bottleneck



2. Gradient field:

$$\vec{F} = \nabla f$$

$$F(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

The vector assigned to each point is directed along the direction of largest rate of change of f, and its magnitude is the directional derivative along this direction.

Line Integral of a Vector Field

Curve C: $\bar{r}(t)$, $a \le t \le b$.

Unit tangent vector: $\hat{T} = \dfrac{\bar{v}(t)}{|\bar{v}(t)|} = \dfrac{d\bar{r}}{ds}$ → book uses **T** (bold T)

Def

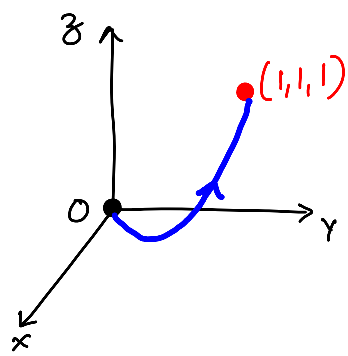
The line integral of $\bar{F}$ along C is $$\int_C \bar{F}\cdot\hat{T}\,ds = \int_C \left(\bar{F}\cdot\frac{d\bar{r}}{ds}\right)ds = \int_C \bar{F}\cdot d\bar{r}.$$

To evaluate, we compute $$\int_a^b \left(\bar{F}(\bar{r}(t))\cdot\left(\frac{d\bar{r}}{dt}\right)\right)dt.$$

7. $\bar{F} = 3y\,\hat{i} + 2x\,\hat{j} + 4z\,\hat{k}$

(b) C_2 : $\bar{r}(t) = \underbrace{t}_{x}\,\hat{i} + \underbrace{t^2}_{y}\,\hat{j} + \underbrace{t^4}_{z}\,\hat{k}$, $0 \le t \le 1$.

Find line integral of vector field F along C_2 from $(0,0,0)$ to $(1,1,1)$.



$$\bar{F}(t) = 3\underbrace{t^2}_{y}\,\hat{i} + 2\underbrace{t}_{x}\,\hat{j} + 4\underbrace{t^4}_{z}\,\hat{k}$$

$$\frac{d\bar{r}}{dt} = 1\hat{i} + 2t\hat{j} + 4t^3\hat{k}$$

$$\int_{C_2} \bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right)dt = \int_0^1 \left(3t^2 + 2t\times 2t + 4t^4\times 4t^3\right)dt = \int_0^1 \left(7t^2 + 16t^7\right)dt$$

$$= \frac{7}{3}t^3 + \frac{16}{8}t^8 \Big|_0^1 = \frac{7}{3} + 2 = \frac{13}{3}.$$

This line integral can be used to evaluate the work done in moving an object along curve C from $\underbrace{t=a}_{\text{pt. A}}$ to $\underbrace{t=b}_{\text{pt B}}$ over a force field $\bar{F}$.

$$W = \int_C \bar{F}\cdot\hat{T}\,ds = \int_a^b \left(\bar{F}(\bar{r}(t))\cdot\frac{d\bar{r}}{dt}\right)dt.$$

(19). $\bar{F} = xy\,\hat{i} + y\,\hat{j} - yz\,\hat{k}$, $\quad \bar{r}(t) = \underbrace{t}_{x}\,\hat{i} + \underbrace{t^2}_{y}\,\hat{j} + \underbrace{t}_{z}\,\hat{k}$, $\quad 0 \le t \le 1$.

Find the work done in moving along C in the direction of increasing t (from $t=0$ to $t=1$).

$$\bar{F}(\bar{r}(t)) = t\,t^2\,\hat{i} + t^2\,\hat{j} - t^2 t\,\hat{k} = t^3\,\hat{i} + t^2\,\hat{j} - t^3\,\hat{k}$$

$$\frac{d\bar{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}. \qquad \bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right) = t^3 + 2t^3 - t^3 = 2t^3$$

$$W = \int_0^1 \bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right)dt = \int_0^1 2t^3\,dt = \frac{2}{4}t^4\Big|_0^1 = \frac{1}{2}.$$