MATH 273 – Lecture 28 (12/09/2014)

Flow and Circulation

If $\bar{F}$ is a continuous velocity field, then the **flow** along the curve C from $A = \bar{r}(a)$ to $B = \bar{r}(b)$ is given by

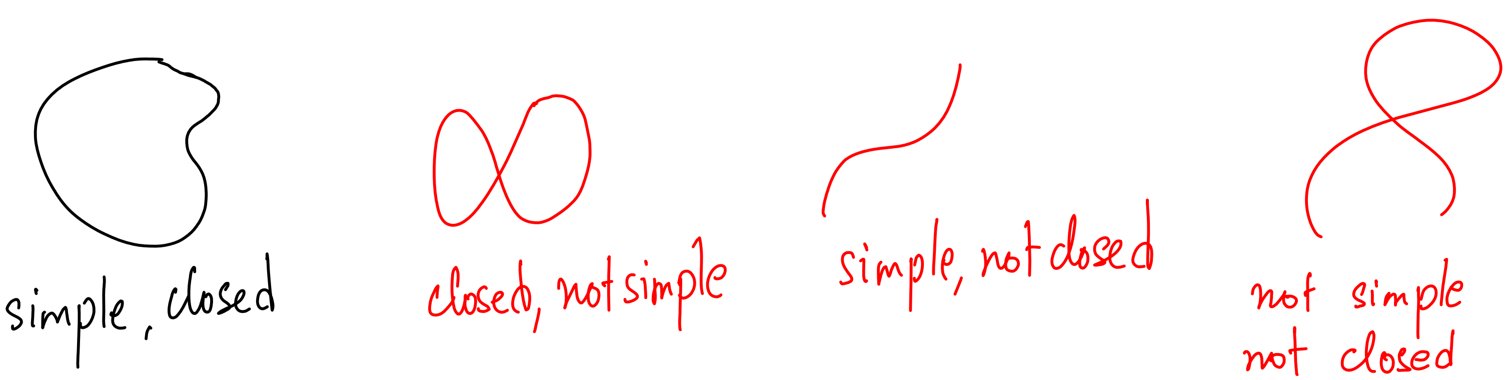
$$\text{Flow} = \int_C F \cdot \hat{T}\, ds. \longrightarrow$$ evaluate it similar to how we compute work.

If $A = B$, C is a closed curve, and the flow is then called the **circulation** around C.

We will consider, in detail, **simple closed** curves

simple → does not cross itself

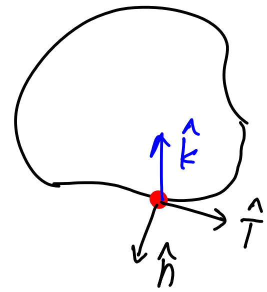
closed → loop



Circulation – adds $\bar{F}\cdot\hat{T}$ over C

We now consider $\bar{F}\cdot\hat{n}$, where $\hat{n}$ is the unit normal vector.

$\hat{k}$, the z-unit vector, points up from the plane, and is perpendicular to both $\hat{T}$ and $\hat{n}$.



Def If C is a smooth simple closed curve in the domain of a vector field $\bar{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ in the plane, and $\hat{n}$ is the outward pointing normal vector on C, then the

$$\textbf{Flux across } C = \int_C \bar{F}\cdot\hat{n}\,ds.$$

How do we compute $\hat{n}$? We orient C in the counter-clockwise (CCW) direction. Then $\hat{n} = \hat{T}\times\hat{k}$ (right-hand rule for cross-product). So

$$\hat{n} = \underbrace{\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right)}_{\hat{T}}\times\hat{k} = \underbrace{\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}}_{\hat{n}}.$$

So $\bar{F}\cdot\hat{n} = (M\hat{i}+N\hat{j})\cdot\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right)$

$$= M\frac{dy}{ds} - N\frac{dx}{ds}.$$

Thus, flux across $C = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds$

i.e., $C = \int_C M\,dy - N\,dx.$

Prob 29. $\bar{F} = x\hat{i} + y\hat{j}$. (a) $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$.

C (unit circle)

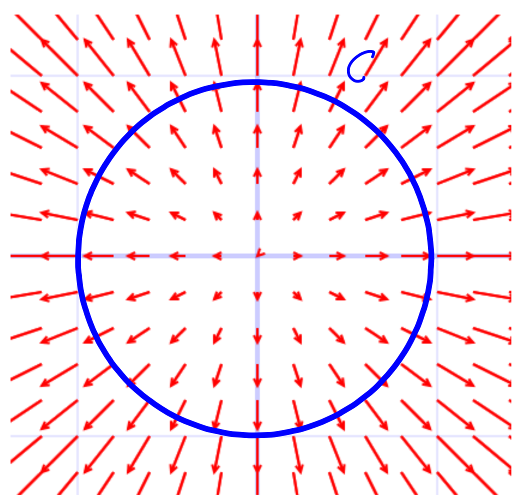
Find circulation of $\bar{F}$ around C, and flux of $\bar{F}$ across C.

$\bar{r}(t) = \underbrace{\cos t}_{x}\,\hat{i} + \underbrace{\sin t}_{y}\,\hat{j}$

$\bar{F} = \underbrace{x}_{M}\,\hat{i} + \underbrace{y}_{N}\,\hat{j}$

$x = \cos t$ $\qquad$ $y = \sin t$

$dx = -\sin t\,dt$ $\qquad$ $dy = \cos t\,dt$



$$\text{Circulation} = \int_C \bar{F}\cdot\hat{T}\,ds = \int_0^{2\pi} \bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right)dt$$

$$\frac{d\bar{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j}, \quad \bar{F} = \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$\bar{F}\cdot\left(\frac{d\bar{r}}{dt}\right) = -\cos t\sin t + \cos t\sin t = 0.$$ Hence circulation $= 0$.

Flux: $\hat{n} = \hat{T}\times\hat{k} = \left(\frac{d\bar{r}}{dt}\right)\times\hat{k} = \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right)\times\hat{k}$

$$= \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$\text{Flux} = \int_C \bar{F}\cdot\hat{n}\,ds = \int_0^{2\pi}(\underbrace{\cos^2 t + \sin^2 t}_{1})\,dt = 2\pi.$$

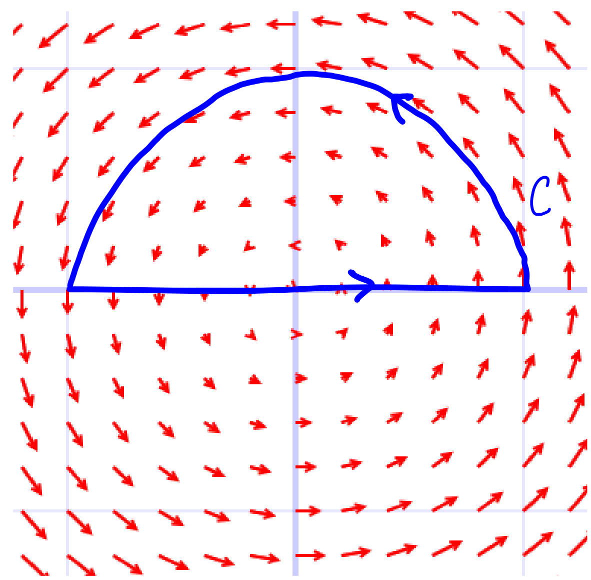
Alternatively, $F = M\hat{i} + N\hat{j}$ where $M = x$, $N = y$

$$\text{Flux} = \int_C M\,dy - N\,dx = \int_0^{2\pi} \underbrace{\cos t}_{M}\cdot\underbrace{\cos t\,dt}_{dy} - \underbrace{\sin t}_{N}\underbrace{(-\sin t\,dt)}_{dx}$$

$$= \int_0^{2\pi}(\cos^2 t + \sin^2 t)\,dt = 2\pi.$$

33. $\bar{F} = -y\hat{i} + x\hat{j}$

C: closed semicircular arch $\bar{r}_1(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$, $0 \le t \le \pi$ followed by the line segment $\bar{r}_2(t) = t\hat{i}$, $-a \le t \le a$.



$\underline{\text{Flow}_1}$: $\bar{r}_1(t) = \underbrace{a\cos t}_{x}\,\hat{i} + \underbrace{a\sin t}_{y}\,\hat{j}$ $\quad dx = -a\sin t\,dt$ $\quad dy = a\cos t\,dt$

$$\bar{F}_1 = \underbrace{-a\sin t}_{M_1}\,\hat{i} + \underbrace{a\cos t}_{N_1}\,\hat{j}$$

$$\frac{d\bar{r}_1}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

$$F_1 \cdot \left(\frac{d\bar{r}_1}{dt}\right) = a^2\sin^2 t + a^2\cos^2 t = a^2$$

$$\text{Flow}_1 = \int_0^{\pi} \bar{F}_1 \cdot \left(\frac{dr}{dt}\right) dt = \int_0^{\pi} a^2\,dt = \pi a^2.$$

$\underline{\text{Flow}_2}$: $\bar{r}_2(t) = \underbrace{t}_{x}\,\hat{i}$, $-a \le t \le a$ $\quad y = 0$

$$\frac{d\bar{r}_2}{dt} = \hat{i}$$

$$\bar{F}_2 = 0\,\hat{i} + t\,\hat{j}$$

$$\bar{F}_2 \cdot \left(\frac{d\bar{r}_2}{dt}\right) = t\hat{j} \cdot \hat{i} = 0.$$

So $\text{Flow}_2 = 0$.

Circulation around $C = \text{Flow}_1 + \text{Flow}_2 = \pi a^2.$

<u>Flux_1</u> $= \int_{C_1} M_1\,dy - N_1\,dx$

$$= \int_0^{\pi} \underbrace{-a\sin t}_{M_1}\,\underbrace{a\cos t\,dt}_{dy} - \underbrace{(a\cos t)}_{N_1}\underbrace{(-a\sin t\,dt)}_{dx} = 0.$$

<u>Flux_2</u> $\quad \bar{r}_2(t) = \underbrace{t}_{x}\,\hat{i} \qquad y=0 \qquad dx = dt$

$\bar{F}_2 = \underbrace{t}_{N_2}\,\hat{j} \qquad M_2 = 0 \qquad dy = 0\,dt = 0$

$$\text{Flux}_2 = \int_{-a}^{a} M_2\,dy - N_2\,dx = \int_{-a}^{a} 0\cdot 0 - t\,dt = \int_{-a}^{a} -t\,dt$$

$$= -\frac{1}{2}t^2\Big|_{-a}^{a} = -\frac{1}{2}\left(a^2 - (-a)^2\right) = 0.$$

$\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0.$