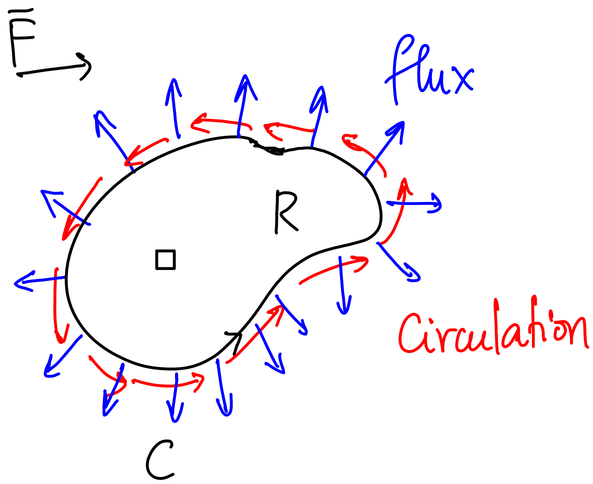
# MATH 273 – Lecture 29 (12/11/2014)

Vector field $\bar{F}$ and a simple closed curve $C$.

Flux across $C = \oint_C \bar{F} \cdot \hat{n}\, ds$.

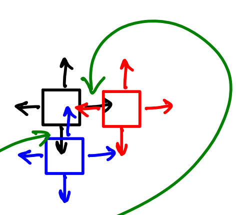



Circulation around $C = \oint_C \bar{F} \cdot \hat{T}\, ds$.

We have seen previously, $\iint_R f(x,y)\, dA$.

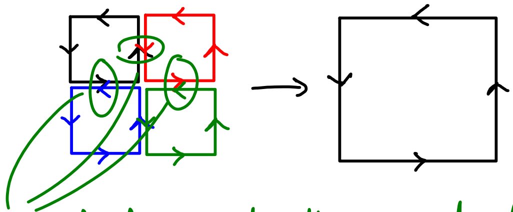
Q: Could we somehow connect these integrals?

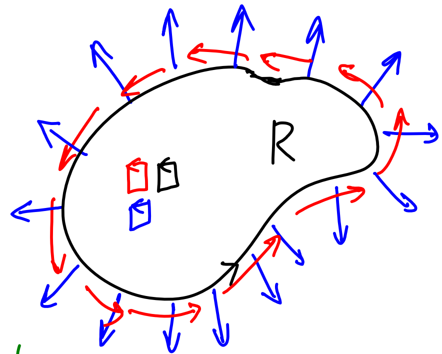
Idea: Consider flux across the boundary of small boxes – $\Delta x$ by $\Delta y$ boxes – in $R$, multiply by area $\Delta A = \Delta x \cdot \Delta y$, and add up the results over all of $R$.



the fluxes at the interfaces cancel each other! Indeed, it appears we should get the total flux when added over all of region $R$.

What about circulation?






circulations at the interface cancel!

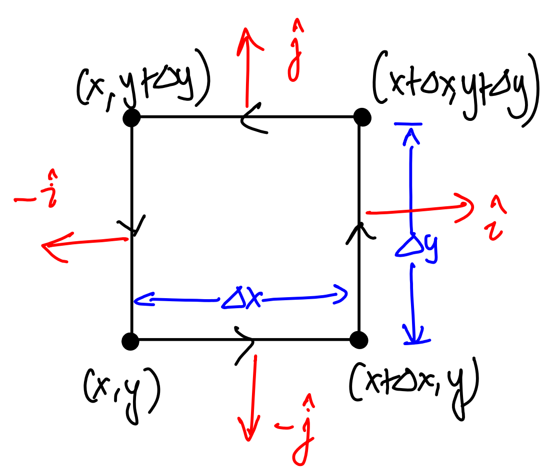
It does seem to work out, in both cases! Green's theorem formalizes this idea. It defines circulation and flux densities at $(x,y)$, which when integrated over $R$ give the circulation and flux.

## Green's theorem in the Plane (Section 15.4)

The vector field is $\bar{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$.

**Def** The **divergence** or **flux density** of the vector field $\bar{F} = M\hat{i} + N\hat{j}$ at $(x,y)$ is

$$\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$




bottom : $\bar{F}\cdot(-\hat{j})\,\Delta x = -N\Delta x$

$$= -N(x,y)\Delta x$$

top : $\bar{F}\cdot(\hat{j})\,\Delta x = N(x, y+\Delta y)\Delta x$

flux across top + bottom $= \underbrace{\Big(N(x,y+\Delta y) - N(x,y)\Big)}_{\frac{\partial N}{\partial y}\cdot\Delta y}\Delta x$

Similarly, we can get that $\frac{\partial N}{\partial y}\cdot \Delta y\,\Delta x$

flux across left + right $= \dfrac{\partial M}{\partial x}\,\Delta x\,\Delta y.$

So, total flux $= \underbrace{\left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right)}_{\text{divergence}}\Delta x\Delta y$

$\Rightarrow$ flux density $= \dfrac{\text{total flux}}{\Delta A} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}.$

Green's theorem (flux-divergence or Normal form)

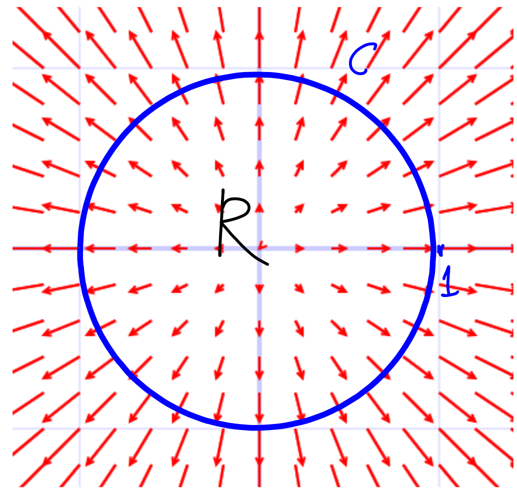
Let $C$ be a piecewise smooth closed curve enclosing region $R$ in the plane. Let $\bar{F} = M\hat{i} + N\hat{j}$, with $M, N$ having continuous first partial derivatives in a open region containing $C$. Then the outward flux of $\bar{F}$ across $C$ is equal to the double integral of div $\bar{F}$ over $R$.

$$\oint_C \bar{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\operatorname{div}\bar{F}} dx\,dy.$$

## Back to Problem 29 (Section 15.2)

$\bar{F} = \underset{M}{x}\hat{i} + \underset{N}{y}\hat{j}$. (a) $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$.

$\to C$ (unit circle)



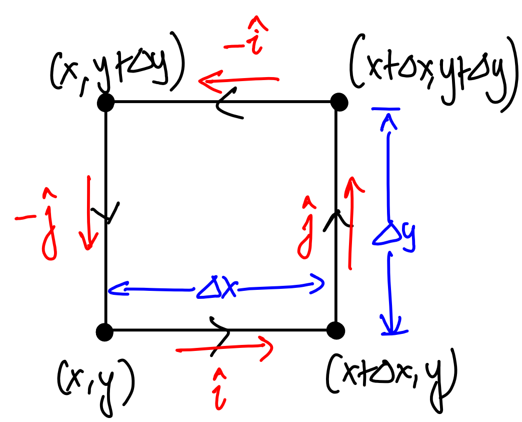

We obtained the flux across $C = 2\pi$.

$$\operatorname{div}\bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \qquad M = x,\ N = y$$

$$= 1 + 1 = 2.$$

$$\oint_C \bar{F}\cdot\hat{n}\,ds = \iint_R 2\,dA = 2(\text{Area}) = 2\pi(1)^2 = 2\pi.$$

## Circulation Density




bottom : $\bar{F}\cdot\hat{i}\,\Delta x = M(x,y)\Delta x$

top : $\bar{F}\cdot(-\hat{i})\Delta x = -M(x, y+\Delta y)\Delta x$

top + bottom : $-\big(M(x, y+\Delta y) - M(x,y)\big)\Delta x$

$$\text{circulation} = -\frac{\partial M}{\partial y}\Delta y\,\Delta x$$

Similarly, for left + right, we get

$$\text{circulation} = \frac{\partial N}{\partial x}\Delta x \Delta y.$$

$$\text{So, total circulation} = \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}} \underbrace{\Delta x \Delta y}_{\text{area}}$$

$$\text{circulation density} = \frac{\text{circulation}}{\text{area}}$$

$$\Rightarrow \textbf{Circulation density} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

also called the k-component of the **curl**, $\text{curl}\, \vec{F} \cdot \hat{k}$.

## Green's theorem (tangential form)

The CCW (counterclockwise) circulation of $\vec{F} = M\hat{i} + N\hat{j}$ around $C$ is equal to the double integral of the circulation density of $\vec{F}$ over $R$.

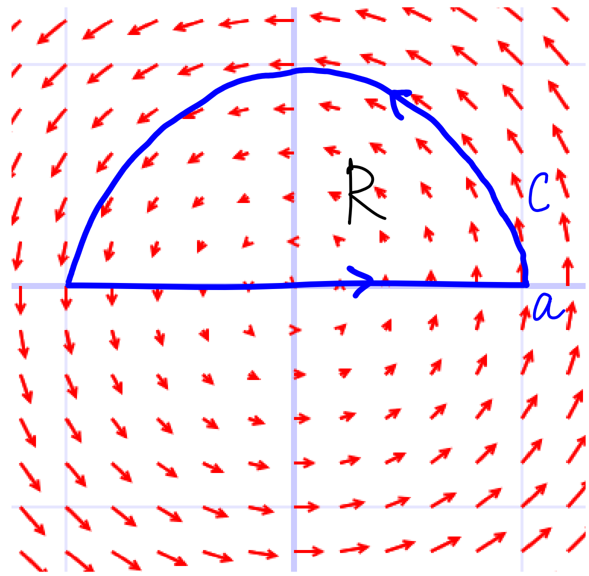
$$\oint_C \vec{F} \cdot \hat{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

Back to problem 33. (Section 15.2)

$$\bar{F} = \underbrace{-y}_{M}\hat{i} + \underbrace{x}_{N}\hat{j}$$

C : closed semicircular arch $\bar{r}_1(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$, $0 \le t \le \pi$ followed by the line segment $\bar{r}_2(t) = t\,\hat{i}$, $-a \le t \le a$.

We had obtained the circulation around $C = \pi a^2$.




$M = -y$, $N = x$

$$\frac{\partial N}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1.$$

Using Green's formula,

$$\oint_C \bar{F} \cdot \hat{T}\, ds = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (1 - -1)\, dA = 2 \iint_R dA = 2\,(\text{Area})$$

$$= 2\left(\frac{1}{2}\pi a^2\right) = \pi a^2.$$

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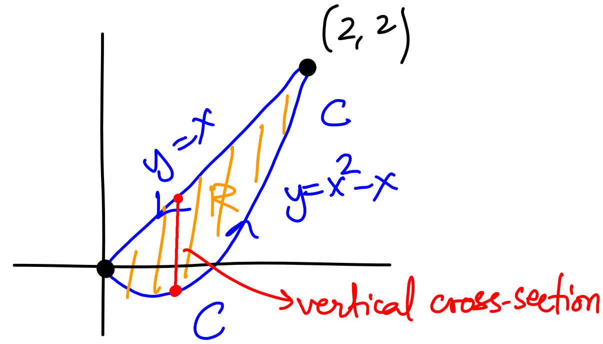
Prob 11, Section 15.4

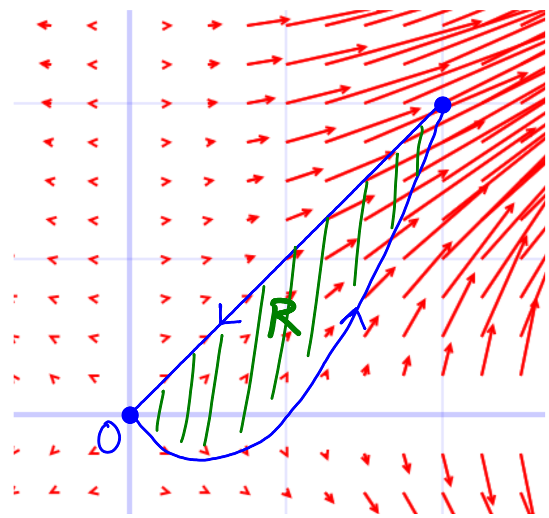
$$\bar{F} = \underbrace{x^3y^2}_{M}\hat{i} + \underbrace{\tfrac{1}{2}x^4y}_{N}\hat{j}$$




Find circulation around and flux across C using Green's theorem.

$$M = x^3y^2 \Rightarrow \frac{\partial M}{\partial x} = 3x^2y^2 \; ; \; \frac{\partial M}{\partial y} = 2x^3y$$

$$N = \tfrac{1}{2}x^4y \Rightarrow \frac{\partial N}{\partial x} = 2x^3y \; ; \; \frac{\partial N}{\partial y} = \tfrac{1}{2}x^4$$




Circulation: $\displaystyle\oint_C \bar{F}\cdot\hat{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$= \iint_R (2x^3y - 2x^3y)\,dA = 0.$$

Flux: $\displaystyle\oint_C \bar{F}\cdot\hat{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

$$= \int_0^2\int_{x^2-x}^{x} \left(3x^2y^2 + \tfrac{1}{2}x^4\right) dy\,dx = \int_0^2 \left(x^2y^3 + \tfrac{1}{2}x^4y \Big|_{\underbrace{x^2-x}_{x(x-1)}}^{x}\right) dx$$

$$= \int_0^2 \left[x^2\left(x^3 - x^3\underbrace{(x-1)^3}_{x^3-3x^2+3x-1}\right) + \tfrac{1}{2}x^4\left(\underbrace{x - x(x-1)}_{2x-x^2}\right)\right] dx$$

$$= \int_0^2 \left[x^2\left(x^3 - x^6 + 3x^5 - 3x^4 + x^3\right) + x^5 - \tfrac{1}{2}x^6\right] dx = \int_0^2 \left[3x^5 - \tfrac{7}{2}x^6 + 3x^7 - x^8\right] dx$$

$$= \tfrac{1}{2}x^6 - \tfrac{1}{2}x^7 + \tfrac{3}{8}x^8 - \frac{x^9}{9}\Big|_0^2 = \underbrace{\frac{(2)^6}{2}}_{32} - \underbrace{\frac{(2)^7}{2}}_{64} + \underbrace{\frac{3}{8}(2)^8}_{3\times 32} - \frac{(2)^9}{9} = 64 - \frac{512}{9} = \frac{64}{9}.$$