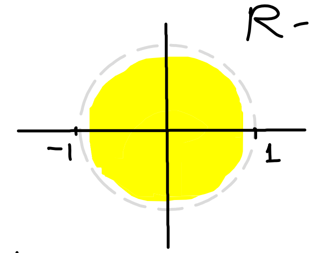
MATH 273 – Lecture 2 (08/28/2014)

Examples of open and closed sets:

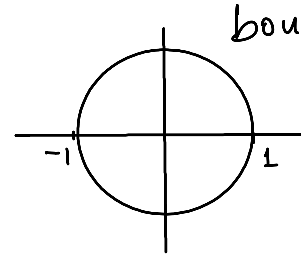
R - open set



The unit disc without the points on the unit circle (shown in dotted gray here) is an open set, as all points in it are interior points.

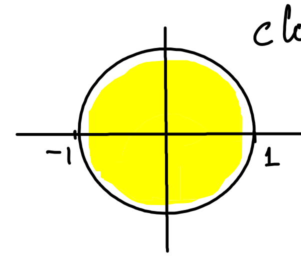
Calling this set R, its boundary is the unit circle.

boundary of R



R and its boundary together form a closed set, which is the closed unit disc.

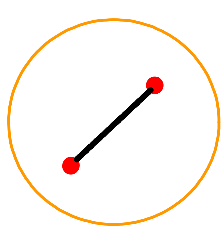
closed set

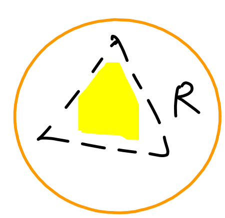


We introduce another concept that captures, in some sense, how large a set or region is, i.e., whether it is limited or unlimited.

Def A region or set is **bounded** if it lies inside a disc of fixed radius. Else it is **unbounded**.

e.g.





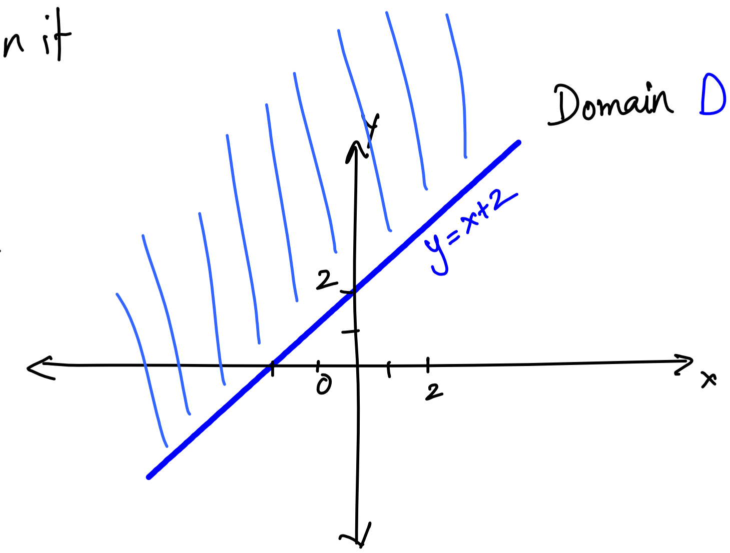
R is open, but is bounded.

A line segment, which is also closed.

Thus, a set can be bounded and at the same time be either open or closed. But notice that if a set is closed, it cannot be unbounded. Why?

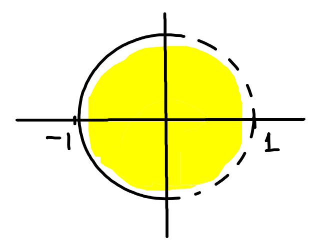
Going back to $f(x,y)=\sqrt{y-x-2}$ from Lecture 1, we can see that its domain D is unbounded.

D is not open here, as all points in it are not interior points. D is actually **closed** here. The line $y=x+2$ is indeed the boundary of D. Since D contains all its boundary points, it is closed.

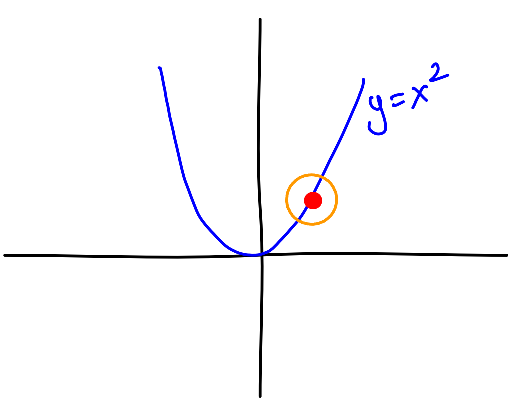


But some sets can be neither open nor closed!

Notice that the set made of the open unit disc and half its boundary is neither open nor closed.



Q. Is the parabola $y=x^2$ open or closed?



Every point on the parabola is a boundary point. So, the set contains all its boundary points. Hence it is closed. Notice that there are no interior points in this set. The set is not open. It is unbounded, though.

How to visualize the range?

Level curves and Surfaces (of functions of two variables)

Def The set of points in the plane where $f(x,y)$ has constant value $f(x,y)=c$ is a **level curve** of f.
The set of all points $(x,y,f(x,y))$ is the **graph** of f, also called the **surface** of f, $z=f(x,y)$.

Prob 15, 21 (13.1)

$f(x,y)=xy$. Draw level curves for $f(x,y)=c$, $c=-9,-1,0,1,9$.

(a) Domain D?
(b) Range?
(c) Describe level curves.
(d) find the boundary of D?
(e) Is D open/closed/neither?
(f) Is D bounded?

$xy=c$

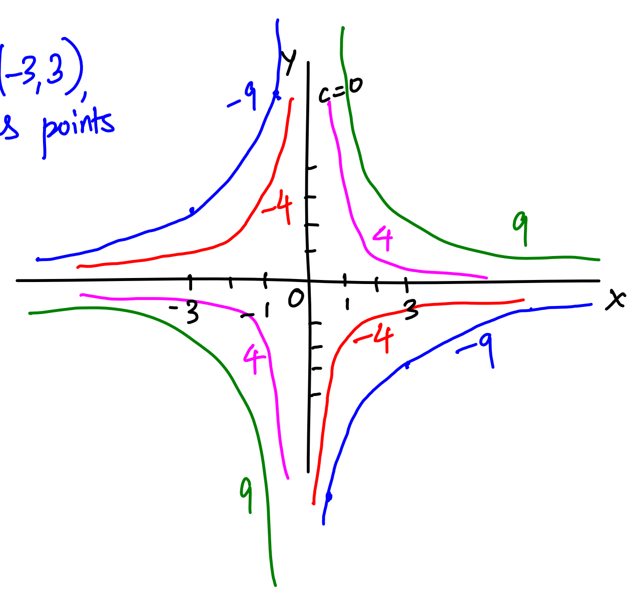
$c=0$, $xy=0$

$c=-9$ $xy=-9$, $y=\frac{-9}{x}$

take $(-1,9)$, $(-3,3)$, and $(-9,1)$ as points

$c=-4$ $xy=-4$, $y=-\frac{4}{x}$

$c=9$, $c=4$



D: entire 2D plane

Range: all real numbers.

D is open and closed (boundary is empty).

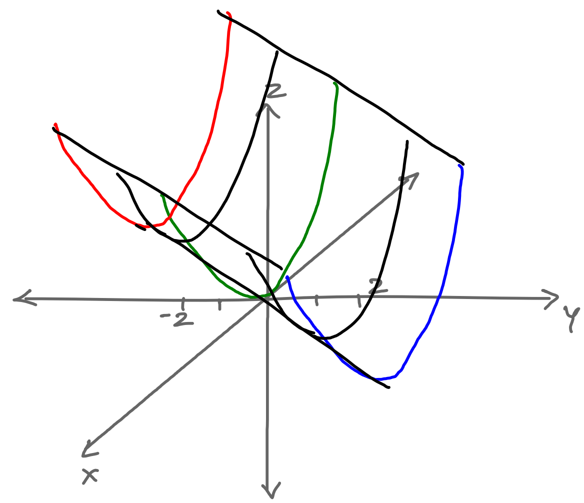
D is unbounded.

41 $f(x,y) = x^2 - y$

plot $(x, y, f(x,y))$.

$z = x^2 - y$

We know how to plot curves in 2D. We plot the 3D surface by plotting several 2D curves together.



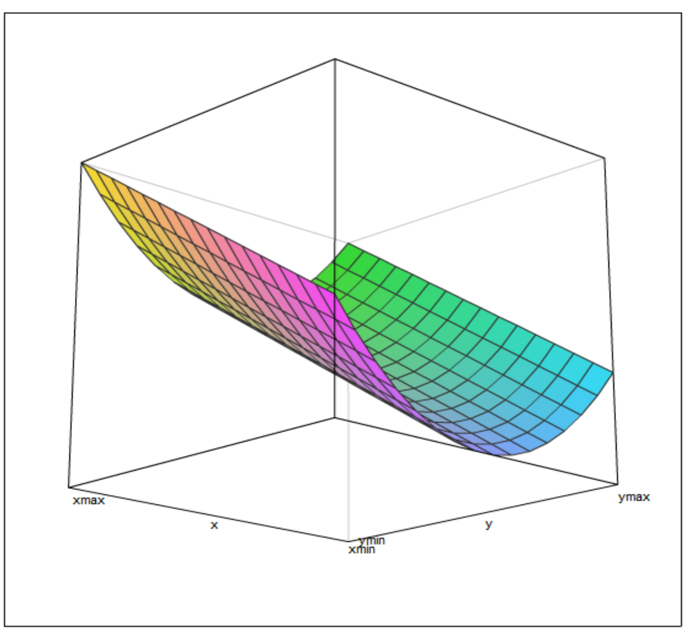
$y=0$: $z = x^2 \longrightarrow$ parabola through origin in xz-plane

$y=2$: $z = x^2 - 2 \rightarrow$ same parabola, but shifted down by 2

$y=-2$: $z = x^2 + 2 \rightarrow$ Same parabola, but shifted up by 2

The "bowl" tilts at a 135° angle for y, as the y term is $-y$.

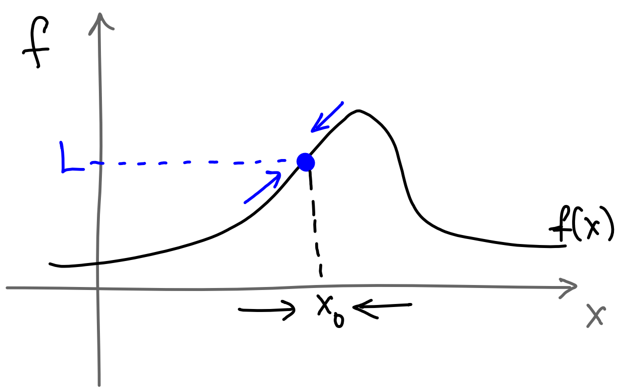
Here is a screenshot of the surface generated on the online 3D grapher (link given in the course web page)



Limits and Continuity in higher dimensions (Section 13.2)

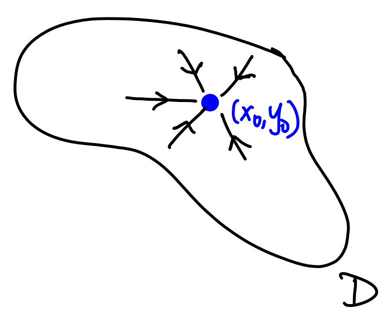
We will **not** cover this section in detail. There will be no homework assigned from this section as well.

Recall the concepts of limits and continuity defined in 1D.



Intuitively, as x approaches x_0 from left or right, if $f(x)$ tends to the same value L, say, then $\lim_{x \to x_0} f(x) = L$.

In 2D, (x,y) could approach some point (x_0,y_0) in infinitely many directions! But we extend the definition of limit the same way – $f(x,y)$ must tend to the same value L as (x,y) approaches (x_0,y_0) from every direction.



The idea of continuity is also extended from 1D to higher dimensions in a similar fashion. $f(x,y)$ is continuous at (x_0,y_0) if

1. $f(x_0,y_0)$ is defined,
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, and
3. the limit in 2. is equal to $f(x_0,y_0)$.

We had defined derivatives of functions as limits (in 1D):

$$\frac{df(x)}{dx} = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}.$$

In multivariable calculus, we essentially apply single variable calculus on one variable at a time, keeping the other variables constant. We get partial derivatives in the latter case.

Partial Derivatives (Section 13.3)

Apply the definition of derivative in 1D to $f(x,y)$ one variable at a time, while keeping the other variable constant.

Def The **partial derivative** of $f(x,y)$ with respect to x at point (x_0, y_0) is

notice $y = y_0$ in both terms

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},$$

also called "partial of f with respect to x", in short

provided that limit exists.

$\frac{\partial f}{\partial x}$ is also denoted $\frac{\partial f(x,y)}{\partial x}$, f_x, and $f_x(x,y)$.

The partial derivative with respect to y is defined analogously.

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h},$$ provided the limit exists.

To find $\frac{\partial f}{\partial x}$, we assume y is constant, and find $\frac{df}{dx}$.

Similarly, to find $\frac{\partial f}{\partial y}$, we assume x is constant, and find $\frac{df}{dy}$.

Prob 3 (13.3, page 711).

$f(x,y) = (x^2-1)(y+2)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = (y+2)\frac{d(x^2-1)}{dx} = (y+2)(2x) = 2x(y+2)$$

y term is constant here

$$\frac{\partial f}{\partial y} = (x^2-1)\frac{d}{dy}(y+2) = (x^2-1)(1) = x^2-1.$$

x-term is constant here