MATH 273 – Lecture 30 (12/14/2014)

Review for the final exam

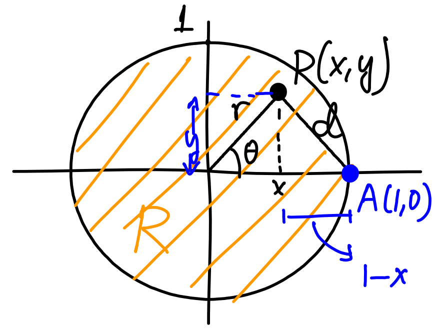
Recall, problems in the final will be from sections covered after Exam II – 14. 14.4, 15.1, 15.2, & Green's theorem evaluation.

5. **(10)** Similar to the definition given in Cartesian (x, y) coordinates, the average value of the function $f(r, \theta)$ over a region R in polar coordinates is given by

$$\hat{f} = \frac{1}{\text{Area}(R)} \iint_R f(r, \theta)\, r dr d\theta.$$

Using the above definition, find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \leq 1$ to the boundary $A(1, 0)$.

$$d^2 = (1-x)^2 + y^2$$
$$= (1 - r\cos\theta)^2 + (r\sin\theta)^2$$
$$= 1 - 2r\cos\theta + \underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{r^2}$$
$$= 1 - 2r\cos\theta + r^2$$



$$\text{Area}(R) = \pi(1)^2 = \pi$$

$$\text{Average squared distance} = \frac{1}{\text{Area}(R)} \iint_R (1 - 2r\cos\theta + r^2)\, r dr d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (1 - 2r\cos\theta + r^2)\, r dr d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{2}{3} r^3 \cos\theta + \frac{r^4}{4} \Big|_0^1 \right) d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\underbrace{\frac{1}{2} - \frac{2}{3}\cos\theta + \frac{1}{4}}_{\frac{3}{4}} \right) d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3}{4} - \frac{2}{3}\cos\theta \right) d\theta = \frac{1}{\pi} \left[\frac{3}{4}\theta - \frac{2}{3}\sin\theta \right] \Big|_0^{2\pi} = \frac{1}{\cancel{\pi}} \frac{3}{\cancel{4}_2} (\cancel{2\pi} - 0) - \frac{2}{3}(0 - 0)$$

$$= \frac{3}{2}.$$

9. **(20)** Find the flux and circulation by evaluating the line integrals (in Part 9a). Then compute these quantities using Green's theorem (in Part 9b).

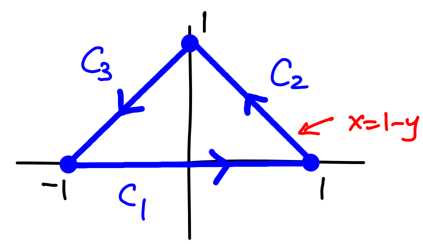
(a) Find the flux across and the circulation around the triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$ of the vector field $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$. Recall that you want the *outward* flux and the *counterclockwise* circulation.

(b) For $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ and a piecewise smooth closed curve C which bounds the region R, two forms of Green's theorem (in 2D) can be specified as follows. Here, $\mathbf{T}$ is the unit tangent and $\mathbf{n}$ is the unit normal vector at each point on C.

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \quad \text{and} \quad \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Find the circulation and flux for the field $\mathbf{F}$ and closed curve C given in Part 9a by evaluating the corresponding double integrals specified by Green's theorem.

$$C_1:\ \bar{r}_1(t) = \underbrace{t}_{x}\,\hat{i}, \quad -1 \le t \le 1.$$



$$\hat{T}_1 = \frac{d\bar{r}_1}{dt} = \hat{i}, \quad \hat{n}_1 = \hat{T}_1 \times \hat{k} = \hat{i} \times \hat{k} = -\hat{j}.$$

$$\bar{F}_1(t) = \underbrace{(t+0)}_{x+y}\hat{i} \underbrace{-}_{-} \underbrace{(t^2+0^2)}_{x^2+y^2}\hat{j} = t\hat{i} - t^2\hat{j}$$

$$\text{Circulation}_1 = \int_{C_1} \bar{F}_1\cdot\hat{T}_1\,ds = \int_{-1}^{1} t\,dt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0.$$

$$\text{Flux}_1 = \int_{C_1} \bar{F}_1\cdot\hat{n}_1\,ds = \int_{-1}^{1} t^2\,dt = \frac{1}{3}t^3\Big|_{-1}^{1} = \frac{1}{3}\left(1^3-(-1)^3\right) = \frac{2}{3}.$$

$$C_2:\ x = 1-y,\ 0 \le y \le 1 \text{ gives } \bar{r}_2(t) = \underbrace{(1-t)}_{x}\,\hat{i} + \underbrace{t}_{y}\,\hat{j}, \quad 0 \le t \le 1.$$

$$\hat{T}_2 = \frac{d\bar{r}_2}{dt} = -\hat{i}+\hat{j}; \quad \hat{n}_2 = \hat{T}_2 \times \hat{k} = \hat{j} + \hat{i} = \hat{i} + \hat{j}.$$

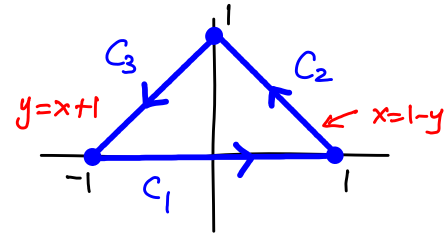
$$\bar{F}_2 = \underbrace{(1-t+t)}_{x+y}\hat{i} \underbrace{-}_{-} \underbrace{\left((1-t)^2+t^2\right)}_{x^2+y^2}\hat{j} = \hat{i} - (2t^2-2t+1)\hat{j}.$$

$$\text{Circulation}_2 = \int_{C_2} \bar{F}_2\cdot\hat{T}_2\,ds = \int_0^1 \Big[\underbrace{-1-(2t^2-2t+1)}_{-2t^2+2t-2}\Big]dt = -\frac{2}{3}t^3 + t^2 - 2t\Big|_0^1$$

$$= -\frac{2}{3} + 1 - 2 = -\frac{5}{3}.$$

$$\text{Flux}_2 = \int_{C_2} \bar{F}_2 \cdot \hat{n}_2 \, ds = \int_0^1 [\underbrace{1-(2t^2-2t+1)}_{-2t^2+2t}] \, dt = -\frac{2}{3}t^3 + t^2 \Big|_0^1 = -\frac{2}{3} + 1 = \frac{1}{3}.$$

$$C_3: \bar{r}_3(t) = \underbrace{-t}_{x}\,\hat{i} + \underbrace{(1-t)}_{y}\,\hat{j}, \quad 0 \le t \le 1.$$



Could also try $-1 \le t \le 0$, with $\bar{r}_3(t) = (-t-1)\hat{i} + -t\hat{j}$.

$$\hat{T}_3 = \frac{d\bar{r}_3}{dt} = -\hat{i} - \hat{j} \; ; \; \hat{n}_3 = \hat{T}_3 \times \hat{k} = \hat{j} - \hat{i} = -\hat{i} + \hat{j}.$$

$$\bar{F}_3(t) = (\underbrace{-t+(1-t)}_{x+y})\hat{i} \underbrace{-}_{-} (\underbrace{(-t)^2 + (1-t)^2}_{x^2+y^2})\hat{j} = (1-2t)\hat{i} - (2t^2-2t+1)\hat{j}.$$

$$\text{Circulation}_3 = \int_{C_3} \bar{F}_3 \cdot \hat{T}_3 \, ds = \int_0^1 [-(\cancel{1}-\cancel{2t}) + (2t^2 - \cancel{2t} + \cancel{1})] \, dt = \int_0^1 2t^2 \, dt = \frac{2}{3}t^3 \Big|_0^1 = \frac{2}{3}.$$

$$\text{Flux}_3 = \int_{C_3} \bar{F}_3 \cdot \hat{n}_3 \, ds = \int_0^1 [\underbrace{-(1-2t)-(2t^2-2t+1)}_{-2t^2+4t-2}] \, dt = -\frac{2}{3}t^3 + 2t^2 - 2t \Big|_0^1 = -\frac{2}{3} + 2 - 2 = \frac{-2}{3}.$$

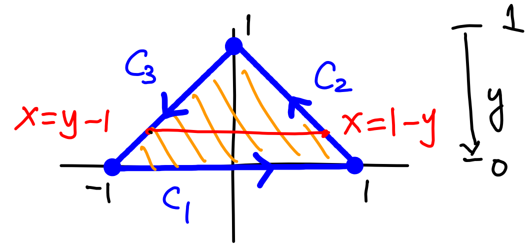
$$\text{Total Circulation} = \text{Circulation}_1 + \text{Circulation}_2 + \text{Circulation}_3$$
$$= 0 - \frac{5}{3} + \frac{2}{3} = -1.$$

$$\text{Total Flux} = \frac{2}{3} + \frac{1}{3} - \frac{2}{3} = \frac{1}{3}.$$

(b) $\bar{F} = \underbrace{(x+y)}_{M}\,\hat{i} - \underbrace{(x^2+y^2)}_{N}\,\hat{j}$ So $M = x+y$, $N = -(x^2+y^2)$.

$$\frac{\partial M}{\partial x} = 1;\quad \frac{\partial M}{\partial y} = 1;\quad \frac{\partial N}{\partial x} = -2x,\quad \frac{\partial N}{\partial y} = -2y$$

$$\text{Circulation} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$



$$= \int_0^1 \int_{y-1}^{1-y} (-2x-1)\,dx\,dy = \int_0^1 \left(-x^2 - x \Big|_{y-1}^{1-y}\right) dy$$

$$= -\int_0^1 \left[\cancel{(1-y)^2} + (1-y) - \cancel{(y-1)^2} - (y-1)\right] = -\int_0^1 (2-2y)\,dy = 2y - y^2 \Big|_1^0$$

$$= 2(0-1) - (0^2 - (1)^2) = -2 + 1 = -1.$$

$$\text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \int_0^1 \int_{y-1}^{1-y} (1-2y)\,dx\,dy = \int_0^1 \left((1-2y)\,x \Big|_{y-1}^{1-y}\right) dy$$

$$= \int_0^1 \underbrace{(1-2y)[1-y-(y-1)]}_{(1-2y)(2-2y)}\,dy = \int_0^1 (4y^2 - 6y + 2)\,dy = \frac{4}{3}y^3 - 3y^2 + 2y \Big|_0^1$$

$$= \frac{4}{3} - 3 + 2 = \frac{1}{3}.$$

30·5

10. **(12)** Decide whether each of the following statements is *True* or *False*. **Justify** your answer.

(a) The value of a line integral of a function along a curve depends only on the values of the function at the end points.

(b) Integration using polar coordinates could be applied only to regions that are circular in shape and bounded by smooth curves.

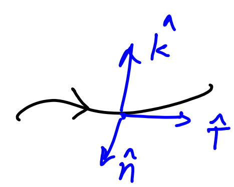
(c) The flux across the region bounded by a simple closed curve changes sign when the direction of traversal of the curve is reversed from counterclockwise to clockwise.

(d) Values of the circulation and the flux of a vector field **F** around and across the region R bounded by a smooth closed curve C can never be equal.

(a) FALSE. The value depends typically on the path taken between the end points.

(b) FALSE. Polar coordinates are an equivalent alternative coordinate system. Its use does not depend on the actual function or region in the integration.

(c) TRUE. The unit normal is reversed at each point by reversing the orientation of the curve.



(d) FALSE. They could be equal. Trivially, consider $\bar{F} = 0\hat{i} + 0\hat{j}$, the zero vector field.